1 Rolling of a hexagonal prism[1]

1.1 Problem text

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil (Figure 1.1). The mass of the prism is M and it is uniformly distributed. The length of each side of the cross-sectional hexagon is a. The moment of inertia I of the hexagonal prism about its central axis is

$$I = \frac{5}{12} M a^2 \tag{1.1}$$

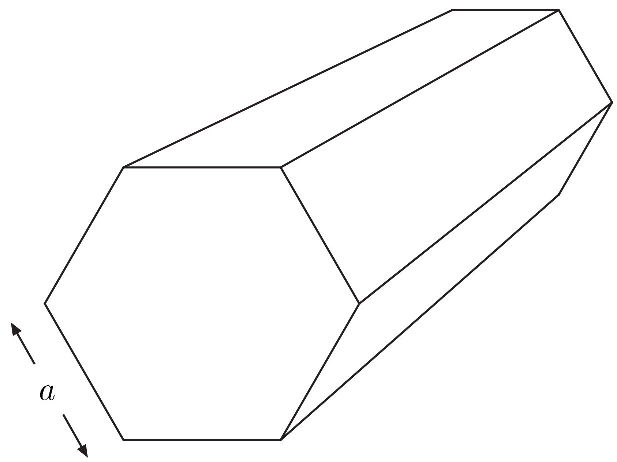

Figure 1.1: *A solid prism with the cross section of a regular hexagon.*

The moment of inertia I' about an edge of the prism is

$$I' = \frac{17}{12} M a^2 \tag{1.2}$$

a) (3.5 points) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle θ with the horizontal (Figure 1.2). Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is ω_i while ω_f is the angular velocity immediately after the impact.

Show that we may write

$$\omega_f = s\omega_i \tag{1.3}$$

and write the value of the coefficient s on the answer sheet.

[1] Authors: Leó Kristjánsson and Thorsteinn Vilhjálmsson

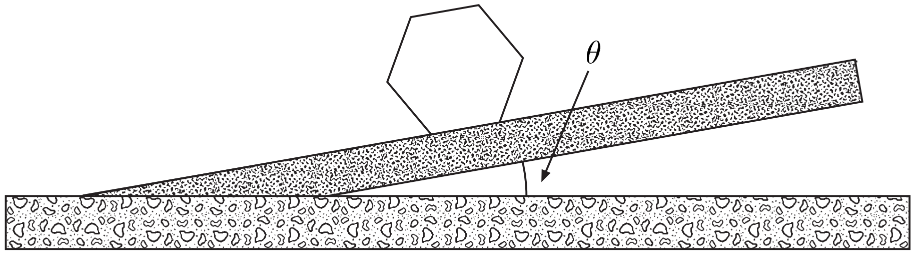

Figure 1.2: *A hexagonal prism lying on an inclined plane.*

b) (1 point) The kinetic energy of the prism just before and after impact is similarly K_i and K_f.

Show that we may write

$$K_f = rK_i \tag{1.4}$$

and write the value of the coefficient r on the answer sheet.

c) (1.5 points) For the next impact to occur K_i must exceed a minimum value $K_{i,min}$ which may be written in the form

$$K_{i,min} = \delta Mga \tag{1.5}$$

where $g = 9.81$ m/s^2 is the acceleration of gravity.

Find the coefficient δ in terms of the slope angle θ and the coefficient r. Write your answer on the answer sheet. (Use the algebraic symbol r, not its value).

d) (2 points) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i,0}$ may be written as:

$$K_{i,0} = \kappa Mga \tag{1.6}$$

and write the coefficient κ in terms of θ and r on the answer sheet.

e) (2 points) Calculate, to within 0.1°, the minimum slope angle θ_0, for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

1.2 Solution

a)
Solution Method 1

At the impact the prism starts rotating about a new axis, i.e. the edge which just hit the plane. The force from the plane has no torque about this axis, so that the angular momentum about the edge is conserved during the brief interval of impact. The linear

2 Water under an ice cap[6]

2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

Density of water:	$\rho_w = 1.000 \cdot 10^3$ kg/m^3
Density of ice:	$\rho_i = 0.917 \cdot 10^3$ kg/m^3
Specific heat of ice:	$c_i = 2.1 \cdot 10^3$ J/(kg °C)
Specific latent heat of ice:	$L_i = 3.4 \cdot 10^5$ J/kg
Density of rock and magma:	$\rho_r = 2.9 \cdot 10^3$ kg/m^3
Specific heat of rock and magma:	$c_r = 700$ J/(kg °C)
Specific latent heat of rock and magma:	$L_r = 4.2 \cdot 10^5$ J/kg
Average outward heat flow through the surface of the earth:	$J_Q = 0.06$ W/m^2
Melting point of ice:	$T_0 = 0$°C, constant

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness d of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle α. The upper surface of the cap slopes by an angle β as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is h_0. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan\alpha, \quad y_2 = h_0 + x \tan\beta \tag{2.1}$$

Derive an expression for the pressure p at the bottom of the ice cap as a function of the horizontal coordinate x and write it on the answer sheet.

Formulate mathematically a condition between β and α, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan\beta = s \tan\alpha$. Find the coefficient s and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8\,x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness h_0 at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line y_1 and add a line y_2 showing the upper surface of the ice. Indicate on the figure which line is which.

[6]Authors: Gudni Axelsson and Thorsteinn Vilhjálmsson

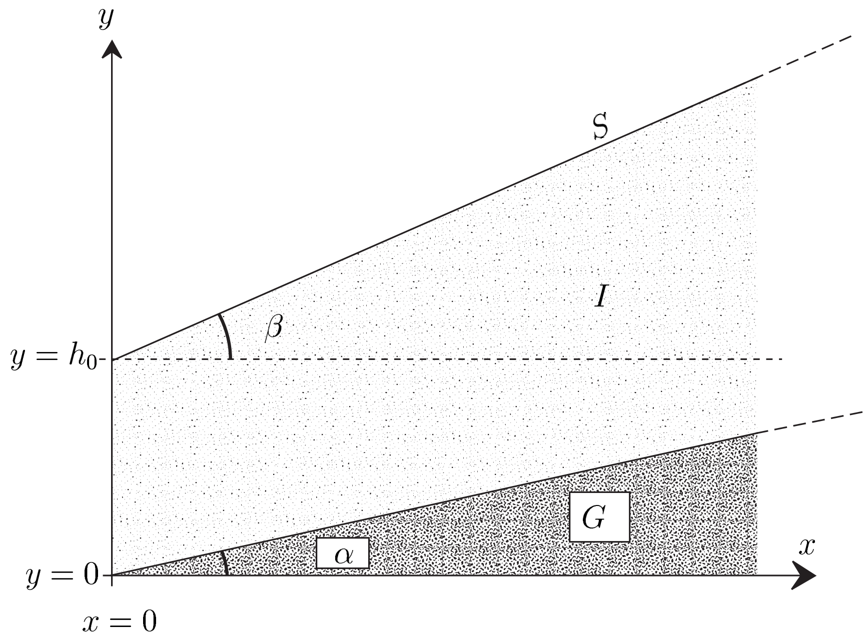

Figure 2.1: *Cross section of an ice cap with a plane surface resting on an inclined plane ground. S: surface, G: ground, I: ice cap.*

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth h of 100 m and a radius r of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C. Subsequently,

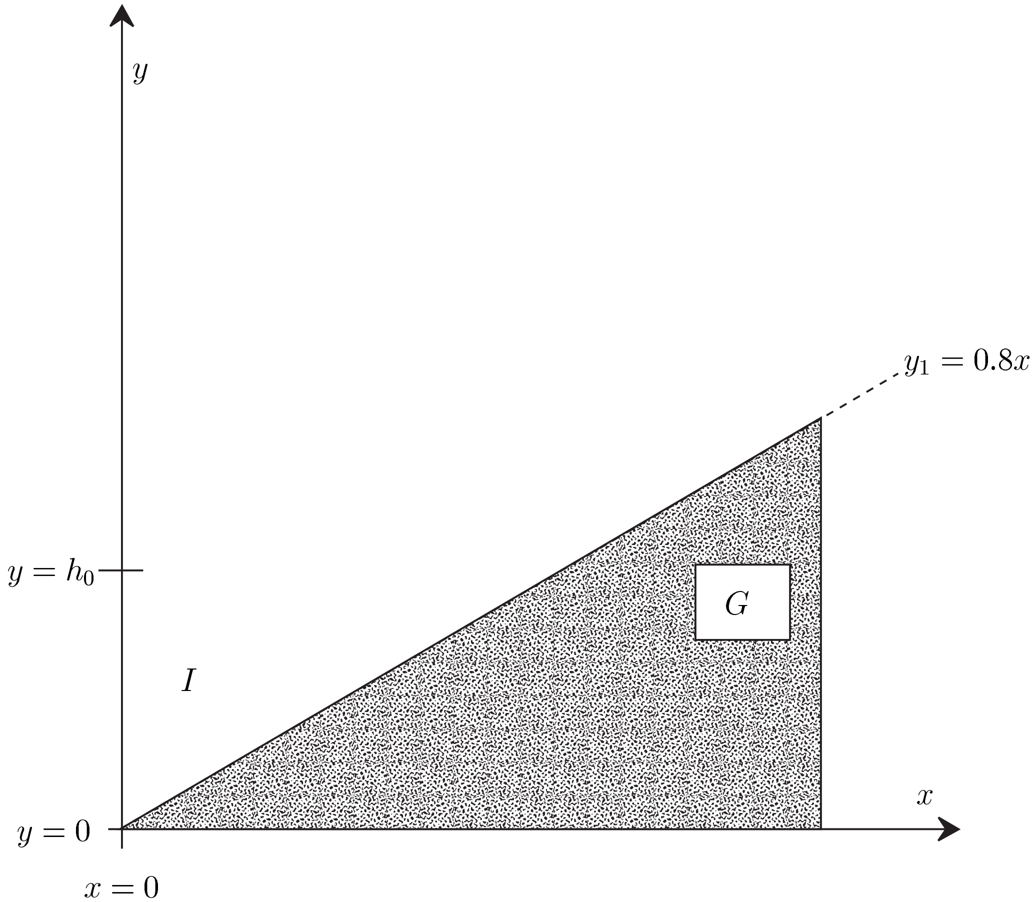

Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

3 Faster than light?[7]

3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is $c = 3.00 \cdot 10^{8}$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and t is the time of observation. The angular speeds, as seen from the Earth, are ω_1 and ω_2. The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of ω_1 and ω_2 in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle ϕ $(0 \leq \phi \leq \pi)$ to the direction towards a distant observer O (Figure 3.2). The speed may be written as $v = \beta c$, where c is the speed of light. The distance to the source, as measured by the observer, is R. The angular speed of the source, as seen from the observer, is ω, and the apparent linear speed perpendicular to the line of sight is $v'_\perp$.

Find ω and $v'_\perp$ in terms of β, R and ϕ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate β and ϕ from the angular speeds ω_1 and ω_2 and the distance R. Here ϕ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for β and ϕ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_\perp$ to be larger than the speed of light c.

[7] Authors: Einar Gudmundsson, Knútur Árnason and Thorsteinn Vilhjálmsson

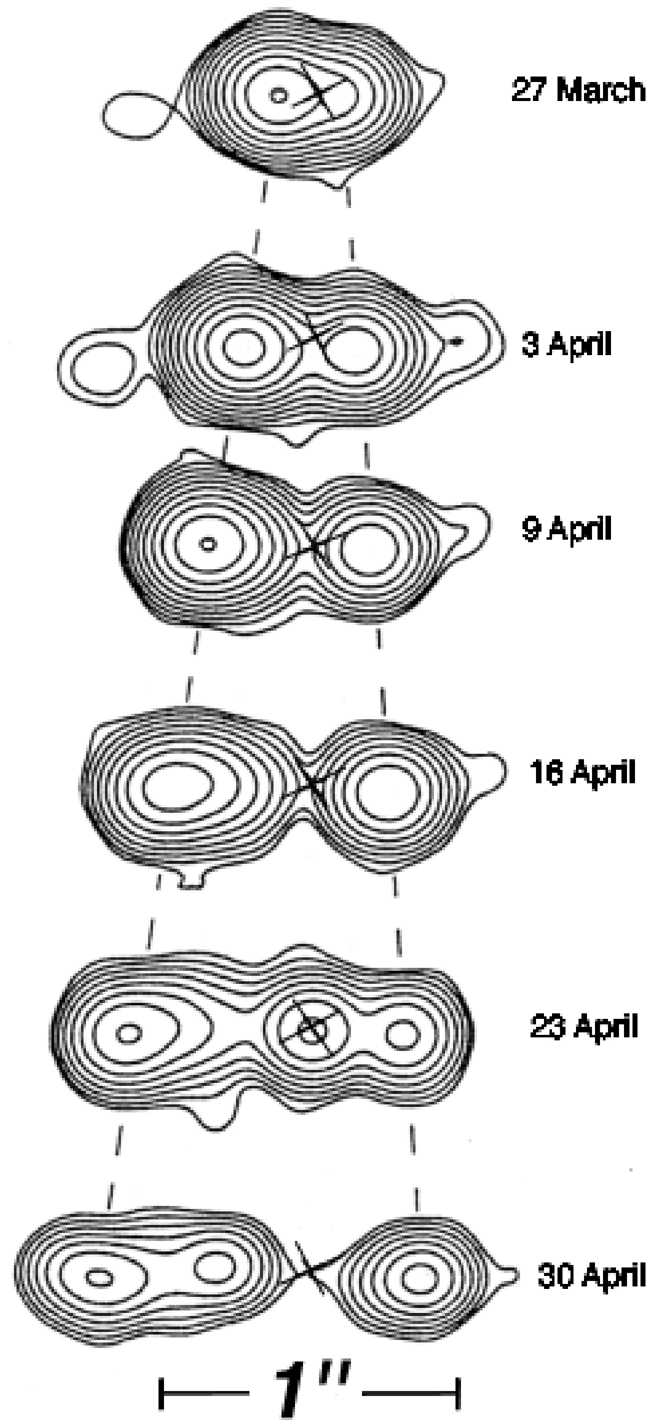

Figure 3.1: *Radio emission from a source in our galaxy.*

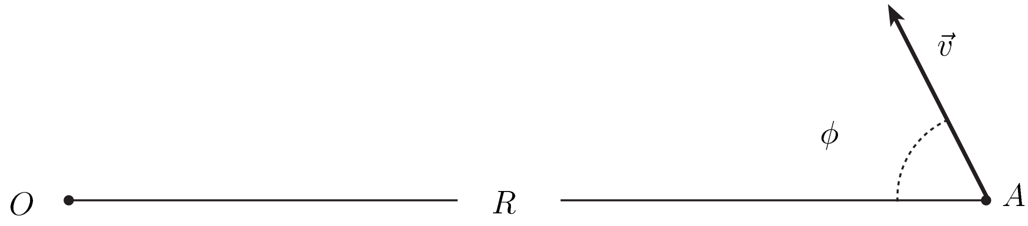

Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda = \lambda_0(1 - \beta\cos\phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\,\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\ . \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data: