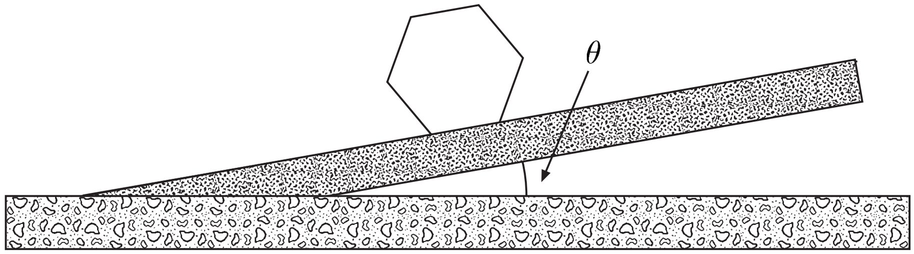


Figure 1.2: *A hexagonal prism lying on an inclined plane.*

b) (1 point) The kinetic energy of the prism just before and after impact is similarly K_i and K_f.

Show that we may write

$$K_f = rK_i \tag{1.4}$$

and write the value of the coefficient r on the answer sheet.

c) (1.5 points) For the next impact to occur K_i must exceed a minimum value $K_{i,min}$ which may be written in the form

$$K_{i,min} = \delta Mga \tag{1.5}$$

where $g = 9.81\ \mathrm{m/s^2}$ is the acceleration of gravity.

Find the coefficient δ in terms of the slope angle θ and the coefficient r. Write your answer on the answer sheet. (Use the algebraic symbol r, not its value).

d) (2 points) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i,0}$ may be written as:

$$K_{i,0} = \kappa Mga \tag{1.6}$$

and write the coefficient κ in terms of θ and r on the answer sheet.

e) (2 points) Calculate, to within 0.1°, the minimum slope angle θ_0, for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

1.2 Solution

a)
Solution Method 1

At the impact the prism starts rotating about a new axis, i.e. the edge which just hit the plane. The force from the plane has no torque about this axis, so that the angular momentum about the edge is conserved during the brief interval of impact. The linear momentum of the prism as a whole has the same direction as the velocity of the center of mass ($\vec{P} = M\ \vec{v}_C$ where the subscript C refers to the center of mass), and this direction is easy to follow when we know the axis of rotation at a given time. Just before impact $\vec{P}$ is directed 30° downwards relative to the plane, but will after impact point 30° upwards from the plane, see Figure 1.3.

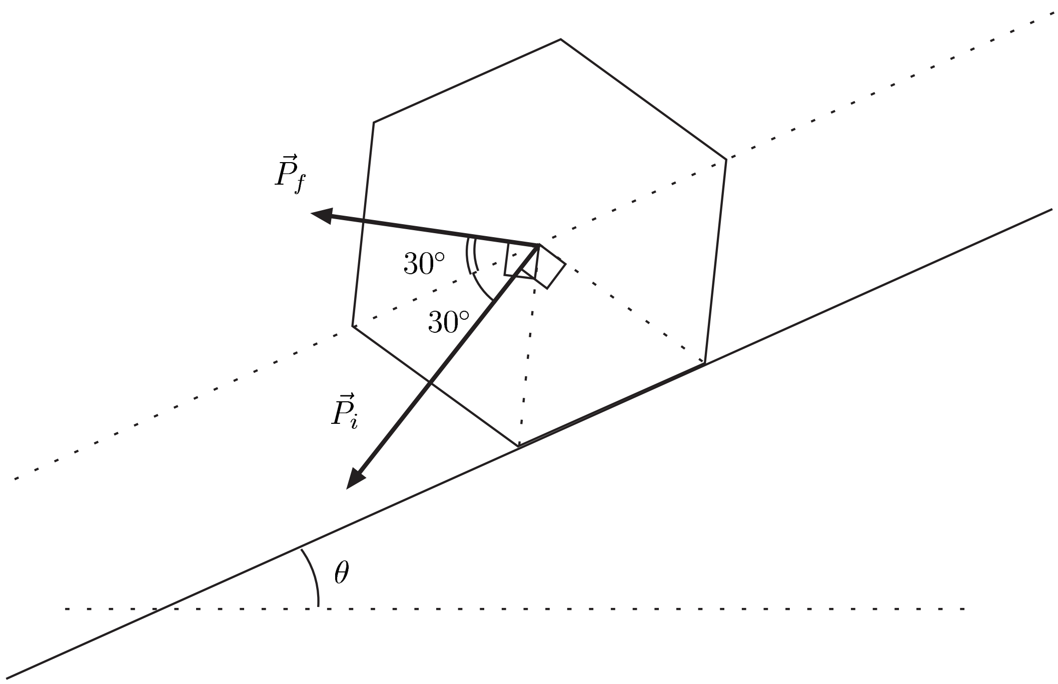


Figure 1.3: *The linear momentum of the prism as a whole, before and after impact.*

To find the angular momentum about the edge of impact just before the impact we use the equation relating angular momentum $\vec{L}$ about an arbitrary axis to the angular momentum $\vec{L}_C$ about an axis through the center of mass parallel to the first one:

$$\vec{L} = \vec{L}_C + M\ \vec{r}_C \times \vec{v}_C \tag{1.7}$$

where the subscript C refers to the center of mass. Here, this is applied to an axis at the point of impact so that $\vec{r}_C$ is the vector from that point to the center of mass (Figure 1.3). The vectors on the right hand side of equation (1.7) both have the same direction. Hence we get for the quantities just before impact[2]

$$|\vec{r}_C \times \vec{v}_{Ci}| = r_C\ v_{Ci} \sin 30° = a^2\ \omega_i\ /\ 2 \tag{1.8}$$

$$L_i = I\ \omega_i + \frac{1}{2}\ M\ a^2\ \omega_i = \left(\frac{5}{12} + \frac{1}{2}\right)\ M\ a^2\ \omega_i = \frac{11}{12}\ M\ a^2\ \omega_i \tag{1.9}$$

On the other hand, angular momentum about the edge just after impact is, from equation (1.2):[3]

[2]This may also be done by using Steiner's theorem twice, going from the previous axis of impact to the center of mass and from there to the new axis of impact.

[3]Alternatively:

$$L_f = I'\omega_f = \frac{17}{12} M a^2 \omega_f \tag{1.10}$$

where the subscript f always refers to the situation just after impact. We may notice that the difference comes about because of the different directions of $\vec{v}_{Ci}$ and $\vec{v}_{Cf}$. Now, when we state the conservation of angular momentum, $L_i = L_f$, we obtain a relation between the angular velocities as follows:

$$\omega_f = \frac{11/12}{17/12}\, \omega_i = \frac{11}{17}\, \omega_i \tag{1.11}$$

We thus get:

$$\boldsymbol{s = 11/17} \tag{1.12}$$

We may note that s is independent of a, ω_i, and θ.

Solution Method 2

On impact the prism receives an impulse $\vec{P}$ [N · s] from the plane at the edge where the impact occurs. There is no reaction at the edge which is leaving the plane. The impulse has a component $P_{||}$ parallel to the inclined plane (positive upwards along the incline in Figure 1.3 and a component $P_{\perp}$ perpendicular to the plane (positive upwards from the plane in the same figure).

We can set up three equations with the three unknowns $P_{||}$, $P_{\perp}$ and the ratio $s = \frac{\omega_f}{\omega_i}$. The quantity $P_{||}$ is the change in the parallel component of the linear momentum of the prism and $P_{\perp}$ is the corresponding change in perpendicular linear momentum. Thus:

$$P_{||} = M\left(\omega_i - \omega_f\right) a \cdot \frac{\sqrt{3}}{2} \tag{1.13}$$

$$P_{\perp} = M\left(\omega_i + \omega_f\right) a \cdot \frac{1}{2}. \tag{1.14}$$

We finally have:

$$P_{\perp} a \frac{1}{2} - P_{||} a \frac{\sqrt{3}}{2} = I\left(\omega_i - \omega_f\right) \tag{1.15}$$

since the right hand side is the change in angular momentum about the center of mass. Equations (1.13), (1.14) and (1.15) can now be solved for the ratio $s = \frac{\omega_f}{\omega_i}$ giving, of course, the same result as before.

$$\begin{aligned} L_f &= I\,\omega_f + M\,|\vec{r}_C \times \vec{v}_{Cf}| = I\omega_f + Ma^2\,\omega_f \sin 90^\circ \\ &= \left(\frac{5}{12} + 1\right) Ma^2\,\omega_f = \frac{17}{12}\,Ma^2\,\omega_f \end{aligned}$$

b)

The linear speed of the center of mass just before impact is $a\omega_i$ and just after impact it is $a\omega_f$. We know that we can always write the kinetic energy of a rotating rigid body as a sum of „internal“ and „external“ kinetic energy:

$$K_{tot} = \frac{1}{2}\ I\ \omega^2 + \frac{1}{2}\ M\ v_C^2 \tag{1.16}$$

From this we see that in our case the kinetic energy K_{tot} is proportional to ω^2 both before and after impact so that we get

$$K_f = r\ K_i = \left(\frac{11}{17}\right)^2 K_i = \frac{121}{289} K_i \tag{1.17}$$

so

$$\boldsymbol{r = 121/289} \approx 0.419 \tag{1.18}$$

c)

The kinetic energy K_f after the impact must be sufficient to lift the center of mass to its highest position, straight above the point of contact. The angle through which $\vec{r}_C$ moves for this is

$$x = \frac{\alpha}{2} - \theta \tag{1.19}$$

where $\alpha = 60°$ is the top angle of the triangles meeting at the center of the polygon.[4] The energy for this lifting of the center of mass is

$$E_0 = Mga(1 - \cos x) = Mga\,(1 - \cos(30° - \theta)) \tag{1.20}$$

and we get the condition

$$K_f = rK_i > E_0 = Mga\,(1 - \cos(30° - \theta)) \tag{1.21}$$

thus

$$\boldsymbol{\delta = \frac{1}{r}\,(1 - \cos(30° - \theta))} \tag{1.22}$$

(Note that $\cos(30° - \theta) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$).

d)

Let $K_{i,n}$ and $K_{f,n}$ be the kinetic energies just before and just after the nth impact. We have shown that we have the relation

[4] In the general case $\alpha = 2\pi/N$.

$$K_{f,n} = r\ K_{i,n} \tag{1.23}$$

where $r = \frac{121}{289}$ for a hexagonal prism. Between subsequent impacts the height of the center of mass of the prism decreases by $a \sin\theta$ and its kinetic energy increases for this reason by

$$\Delta = Mga \sin\theta \tag{1.24}$$

We therefore have

$$K_{i,n+1} = rK_{i,n} + \Delta. \tag{1.25}$$

One does not have to write out the complete expression $K_{i,n}$ as a function of $K_{i,1}$ and n to find the limit. This would actually be a proof that the limit exists (see below) but this is given in the problem text. Hence one can make $K_{i,n+1} \approx K_{i,n}$ arbitrarily accurate for sufficiently large n. The limit $K_{i,0}$ must thus satisfy the iterative formula, i.e.

$$K_{i,0} = rK_{i,0} + \Delta \tag{1.26}$$

yielding the solution

$$K_{i,0} = \frac{\Delta}{1-r}. \tag{1.27}$$

i.e.

$$\boldsymbol{\kappa = \frac{\sin\theta}{1-r}} \tag{1.28}$$

We can also solve the problem explicitly by writing out the full expressions:

$$K_{i,2} = r\ K_{i,1} + \Delta \tag{1.29}$$

$$K_{i,3} = r\ K_{i,2} + \Delta = r^2 K_{i,1} + (1+r)\Delta \tag{1.30}$$

$$\ldots$$

$$K_{i,n} = r^{n-1}K_{i,1} + (1 + r + \ldots + r^{n-2})\Delta \tag{1.31}$$

$$= r^{n-1}K_{i,1} + \frac{1-r^{n-1}}{1-r}\Delta \tag{1.32}$$

In the limit of $n \to \infty$ we get

$$K_{i,n} \to K_{i,0} = \frac{\Delta}{1-r} \tag{1.33}$$

which is, of course, the same result as before.

If we calculate the change in kinetic energy through a whole cycle, i.e. from just before impact number n until just before impact $n+1$ we get

$$\Delta K_{i,n} = K_{i,n+1} - K_{i,n} = (r-1)r^{n-1}K_{i,1} + r^{n-1}\Delta \tag{1.34}$$

$$= r^{n-1}\left(\Delta - (1-r)K_{i,1}\right) \tag{1.35}$$

This is positive if the initial value $K_{i,1} < K_{i,0}$ so that $K_{i,n}$ will then increase up to the limit value $K_{i,0}$. If, on the other hand, $K_{i,1} > K_{i,0}$, the kinetic energy $K_{i,n}$ just before impact will decrease down to the limit $K_{i,0}$.

All of this may remind you of motion with friction which increases with speed. Mathematically speaking, the main difference is that we here are dealing with difference equations instead of differential equations.

e)

For indefinite continuation the limit value of K_i in part (d) must be larger than the minimum value for continuation found in part (c):

$$\frac{1}{1-r}\Delta = \frac{1}{1-r} Mga \sin\theta > Mga\left(1 - \cos(30^\circ - \theta)\right)/r \tag{1.36}$$

We put $A = \frac{r}{1-r} = \frac{121}{168}$:

$$A \sin\theta > 1 - \cos 30^\circ \cos\theta - \sin 30^\circ \sin\theta \tag{1.37}$$

$$(A + 1/2)\sin\theta + \sqrt{3}/2 \cos\theta > 1 \tag{1.38}$$

To solve this we define[5]

$$u = \arccos\left(\frac{A + 1/2}{\sqrt{(A+1/2)^2 + 3/4}}\right) \approx 35.36^\circ \tag{1.39}$$

and obtain

$$\cos u \sin\theta + \sin u \cos\theta > 1/\sqrt{(A+1/2)^2 + 3/4} \tag{1.40}$$

$$\sin(u + \theta) > 1/\sqrt{(A+1/2)^2 + 3/4} \tag{1.41}$$

$$\theta > \arcsin\{1/\sqrt{(A+1/2)^2 + 3/4}\} - u \approx 41.94^\circ - 35.36^\circ = 6.58^\circ \tag{1.42}$$

That is

$$\boldsymbol{\theta_0 \approx 6.58^\circ} \tag{1.43}$$

If $\theta > \theta_0$ *and* the kinetic energy before the first impact is sufficient according to part (c), we will, under the assumptions made, get an indefinite "rolling".

[5]You can of course solve any of the inequalities in a purely numerical way, e.g. by progressive guessing or by using the approximations $\sin\phi \approx \phi$ and $\cos\phi \approx 1 - \phi^2/2$.

1.3 Grading scheme

Part	Points
Part **2(a)**	
Answer: $s = \omega_f/\omega_i = 11/17$, equation (1.12)	**3.5**
Part **2(b)**	
Answer: $r = K_f/K_i = s^2 = 121/289$, equation (1.18)	**1.0**
Part **2(c)**	
Answer: $K_{i,min}$ by δ, equation (1.22)	**1.5**
Part **2(d)**	
Answer: Limit $K_{i,0}$ by $\kappa = \sin\theta/(1-r)$, equation (1.28)	**2.0**
Part **2(e)**	
Answer: Minimum angle $\theta_0 = 6.58°$, equation (1.43)	**2.0**

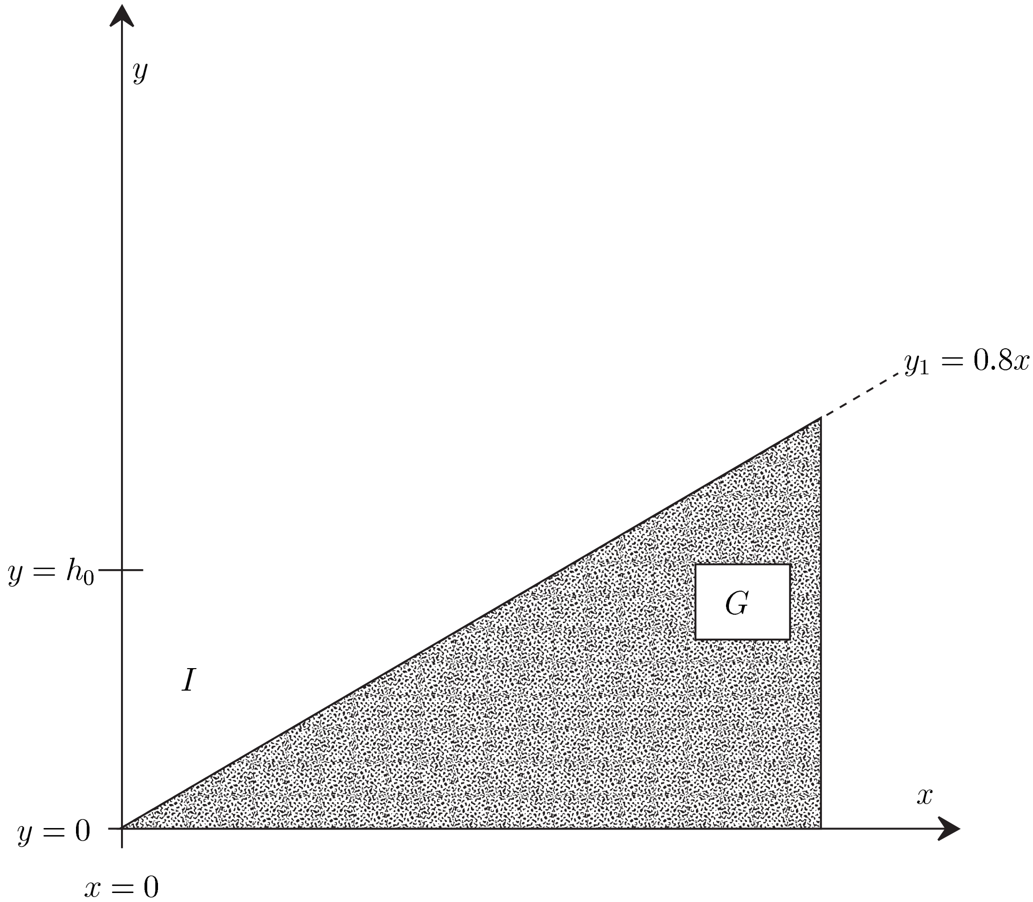


Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

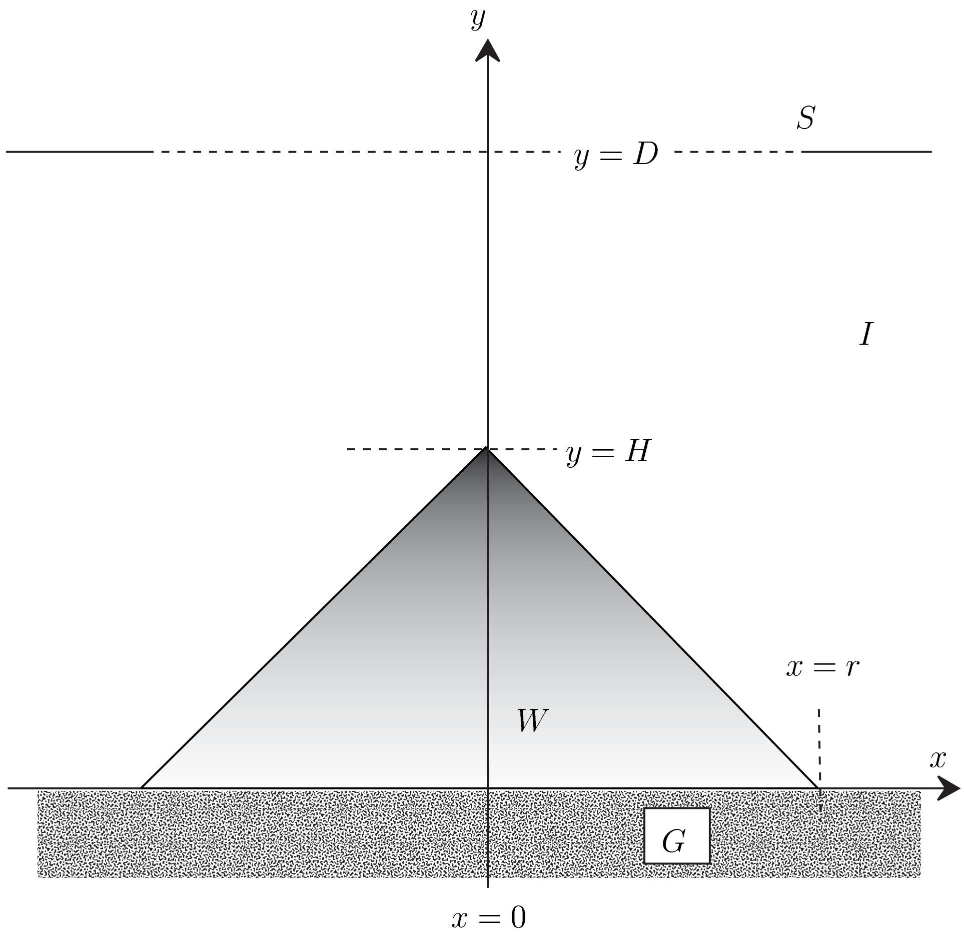


Figure 2.3: *A vertical section through the mid-plane of a water cone inside an ice cap. S: surface, W: water, G: ground, I: ice cap.*

$$\boldsymbol{d} = \frac{J_Q \cdot 1 \text{ year}}{L_i \; \rho_i} = \frac{0.06 \text{ J s}^{-1} \text{ m}^{-2} \; 365.25 \;\; \cdot 24 \;\; \cdot 60 \;\; \cdot 60 \text{ s}}{3.4 \cdot 10^5 \text{ J/kg } 917 \text{ kg/m}^3} = \mathbf{6.1 \cdot 10^{-3} m} \qquad (2.3)$$

b)

Let p_a be the atmospheric pressure, taken to be constant. At a depth z inside the ice cap the pressure is given by:

$$p = \rho_i g z + p_a \qquad (2.4)$$

Therefore, at the bottom of the ice cap, where $z = y_2 - y_1$:

$$\begin{aligned} \boldsymbol{p} &= \rho_i g (y_2 - y_1) + p_a & (2.5) \\ &= \boldsymbol{\rho_i g x (\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a} & (2.6) \end{aligned}$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.

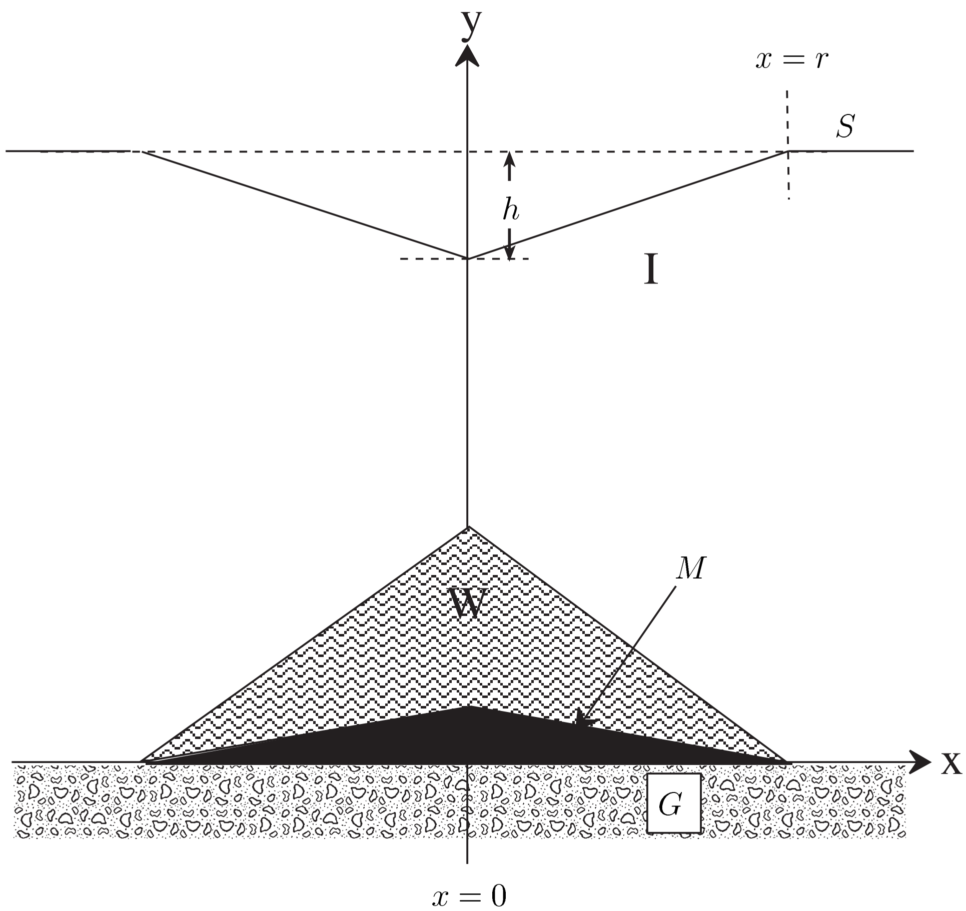


Figure 2.4: *A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, G: ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.*

$$p = constant - \rho_w g y_1 \quad (2.7)$$
$$= constant - \rho_w g x \tan\alpha \quad (2.8)$$

Therefore

$$\rho_i g x(\tan\beta - \tan\alpha) = -\rho_w g x \tan\alpha \quad (2.9)$$

leading to

$$\boldsymbol{\tan\beta} = -\frac{\rho_w - \rho_i}{\rho_i}\ \tan\alpha = \boldsymbol{-\frac{\Delta\rho}{\rho_i}\ \tan\alpha} \approx -0.091\ \tan\alpha \quad (2.10)$$
$$\boldsymbol{s = -\Delta\rho/\rho_i = -0.091} \quad (2.11)$$
$$(2.12)$$

where the minus-sign is significant.

This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. – We now proceed with the solution.

With $\tan\alpha = 0.8$, we get $\tan\beta = -0.073$ and

$$\boldsymbol{y_2 = 2\ \mathbf{km} - 0.073\ x} \tag{2.13}$$

The students are supposed to draw this line on a graph.

c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$\begin{aligned} h &= |r\tan\beta| = \frac{\Delta\rho}{\rho_i} r\tan\alpha && (2.14)\\ &= \frac{\Delta\rho}{\rho_i} H && (2.15)\\ &= 0.091\ \cdot 1\ \text{km} = 91\ \text{m}. && (2.16) \end{aligned}$$

The students are supposed to show this result as a graph.

d)

The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$. We assume that the height of the intrusion is h_1. We may say that it firstly melts an ice cone of its own volume $V_1 = \frac{1}{3}\pi r^2 h_1$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to 0°C the intrusion will further melt a volume equivalent to a cone of height h_3, its water accumulating in place, forming a cone of height $h_3' = \frac{\rho_i}{\rho_w}\ h_3$ relative to the top of the intrusion. The total height of the ice cone melted is

$$h_{tot} = h_1 + h_2 + h_3 \tag{2.17}$$

The depth of the depression at the surface will be given by

$$h = \frac{\Delta\rho}{\rho_i}\ (h_1 + h_3') \tag{2.18}$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$\boldsymbol{H} = h_1 + h_3' = \frac{\rho_i}{\Delta\rho}\ h = \mathbf{1.1 \times 10^3 m} \tag{2.19}$$

The heat balance gives

$$\frac{1}{3}\ \pi\ r^2\ \{\rho_r h_1 (L_r\ + c_r\ \Delta T) - \rho_i\ L_i\ h_{tot}\} = 0 \tag{2.20}$$

where $\Delta T = 1200°C$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ and $h_3 = \frac{\rho_w}{\rho_i} h_3'$ we obtain

$$h_{tot} = h_1 + \frac{\Delta\rho}{\rho_i} h_1 + \frac{\rho_w}{\rho_i} h_3' = \frac{\rho_w}{\rho_i}\,(h_1 + h_3') \tag{2.21}$$

Therefore (using equation (2.19))

$$h_{tot} = \frac{\rho_w}{\rho_i}\,(h_1 + h_3') = \frac{\rho_w}{\rho_i}\,H = \frac{\rho_w}{\Delta\rho}\,h = 1.20 \cdot 10^3 \mathrm{m} \tag{2.22}$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for h_1:

$$\rho_r\,h_1\,(L_r + \, c_r\,\Delta T) = \frac{\rho_i\,\rho_w\,L_i\,h}{\Delta\rho} \tag{2.23}$$

$$\boldsymbol{h_1} = \frac{\rho_i\,\rho_w\,L_i\,h}{\Delta\rho\,\rho_r\,(L_r + \, c_r\,\Delta T)} \tag{2.24}$$

$$= \mathbf{103\ m} \tag{2.25}$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$\boldsymbol{m_{tot}} = \rho_i\,(1/3)\,\pi\,r^2\,h_{tot} = \mathbf{2.9 \cdot 10^{11}\ kg} \tag{2.26}$$

The mass of the water which flows away is

$$\boldsymbol{m'} = \frac{h_1 + h_2}{h_{tot}}\,m_{tot} = \frac{\rho_w\,h_1}{\rho_i\,h_{tot}}\,m_{tot} = \mathbf{2.7 \cdot 10^{10}\ kg} \tag{2.27}$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

2.3 Grading scheme

2(a)	
Answer: equation (2.3), $d = 6.1 \cdot 10^{-3}$ m	**0.5**

2(b)	
Answer i): equation (2.6): $p = \rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a$	**1.0**
Answer ii): equation (2.10): $s = -\frac{\rho_w - \rho_i}{\rho_i} = -\frac{\Delta\rho}{\rho_i}$	**2.0**
Answer iii): Graph based on equation (2.13)	**0.5**

2(c)	
Answer: Depth, radius and graph, $r = 1000$ m, $h = 91$ m	**1.0**

2(d)	
Answer i): Height of water cone as in (2.19): $H = 1.1 \cdot 10^3$ m	**2.0**
Answer ii): Height of intrusion as in (2.25): $h_1 = 103$ m	**1.0**
Answer iii): Total mass of melt water as in (2.26): $m_{tot} = 2.9 \cdot 10^{11}$ kg	**0.5**
Answer iv): Mass of water flowing away as in (2.27): $m' = 2.7 \cdot 10^{10}$ kg	**1.0**
Answer v): Graph	**0.5**

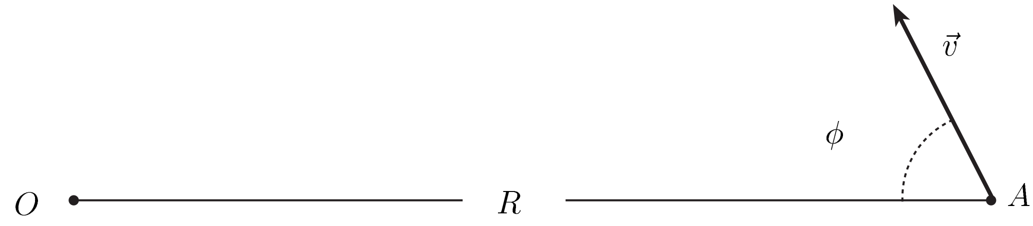


Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift,
$\lambda = \lambda_0(1 - \beta\cos\phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\,\lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\,. \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

time [days]	θ_1 [as]	θ_2 [as]
0	0.139	0.076
7	0.253	0.139
13	0.354	0.190
20	0.468	0.253
27	0.601	0.316
34	0.709	0.367

The uncertainty in the readings by the ruler is estimated to be ± 0.5 mm, resulting in the uncertainty of $\pm$ 0.013 as in the θ values. We plot the data in Figure 3.3.

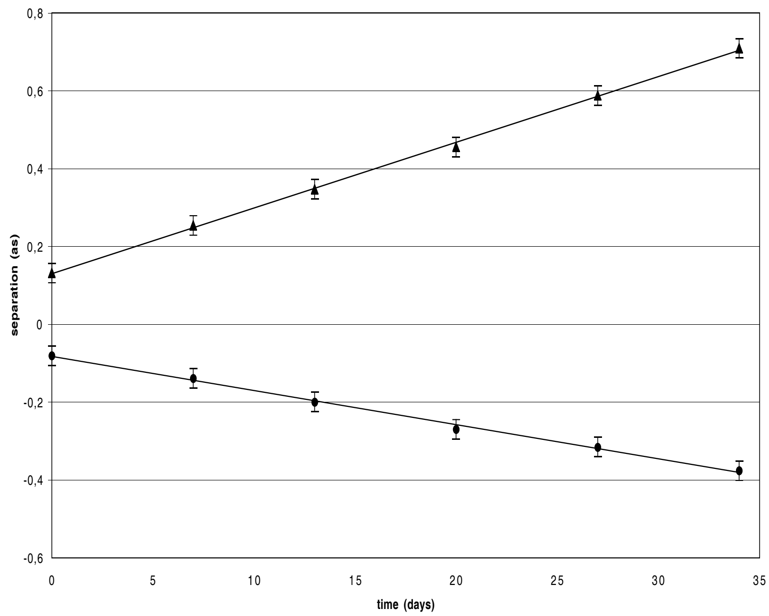


Figure 3.3: *The angular distances θ_1 and θ_2 (in as) as functions of the time in days.*

Fitting straight lines through the data results in:

$$\boldsymbol{\omega_1} = \mathrm{d}\theta_1/\mathrm{dt} = \mathbf{(17.0 \pm 1.0)\ mas/day} = 9.54 \cdot 10^{-13}\mathrm{rad/s} \tag{3.2}$$

$$\boldsymbol{\omega_2} = \mathrm{d}\theta_2/\mathrm{dt} = \mathbf{(8.7 \pm 1.0)\ mas/day} = 4.88 \cdot 10^{-13}\mathrm{rad/s} \tag{3.3}$$

$$\boldsymbol{v'_{1,\perp}} = \omega_1\ R = 9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19} \tag{3.4}$$

$$= \mathbf{3.68 \cdot 10^8 m/s \approx (1.23 \pm 0.07)}\ \boldsymbol{c} \tag{3.5}$$

$$\boldsymbol{v'_{2,\perp}} = \mathbf{1.89 \cdot 10^8 m/s \approx (0.63 \pm 0.07)}\ \boldsymbol{c} \tag{3.6}$$

b) We consider the motion of the source during the time interval Δt from the point A to the point A', see Figure 3.4.

We then have

$$\vec{r}_{AA'} = \vec{r}_{A'} - \vec{r}_A = \vec{v} \cdot \Delta t\ . \tag{3.7}$$

Now let $\Delta t'$ denote the difference in arrival times at O of the signals from A and A'. Due to the different distances to A and A' and the finite speed of light, c, we have

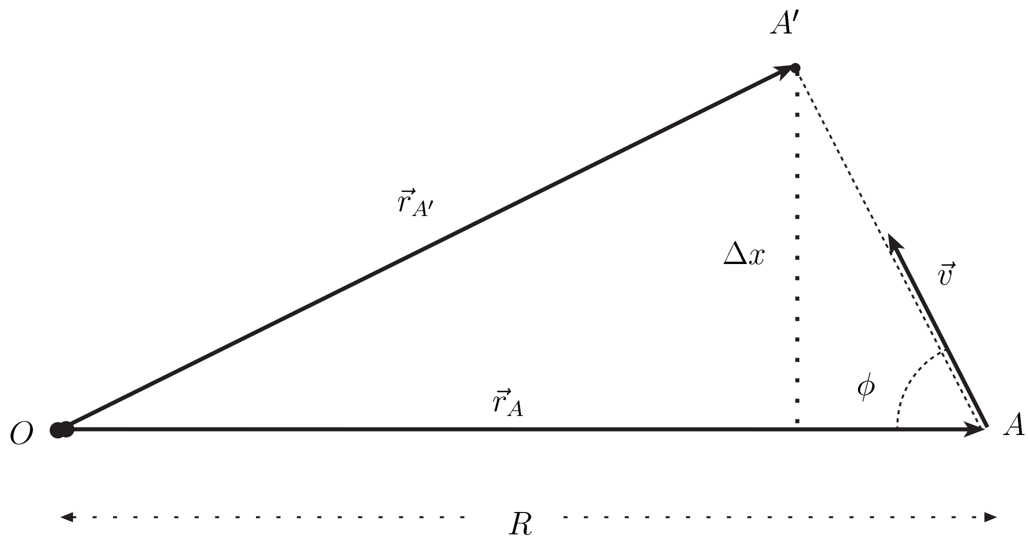


Figure 3.4: *The observer is at O and the original position of the source is at A. The velocity vector is $\vec{v}$.*

$$\Delta t' = \Delta t + (r_{A'} - r_A)/c \; . \tag{3.8}$$

For small Δt, such that $v\ \Delta t << r_A = R$, we have

$$r_{A'} - r_A \approx -v\ \Delta t \cos\phi \tag{3.9}$$

and hence

$$\Delta t' \approx \Delta t\ (1 - \beta\cos\phi)\ ;\ \beta = v/c \; . \tag{3.10}$$

This implies that an observer at O will find the apparent transverse speed of the source to be

$$\boldsymbol{v'_\perp} = \frac{\Delta x}{\Delta t'} = \frac{\Delta x}{\Delta t\ (1-\beta\cos\phi)} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{1-\beta\cos\phi}} \tag{3.11}$$

where we have used that the real transverse speed in the reference frame of the observer is $v_\perp = \Delta x/\Delta t = c\beta\sin\phi$.

The angular speed observed at O is

$$\boldsymbol{\omega} = \frac{v'_\perp}{R} = \frac{\boldsymbol{c\beta\sin\phi}}{\boldsymbol{R\ (1-\beta\cos\phi)}} \tag{3.12}$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi = \phi_1$ we have $\sin\phi_2 = \sin\phi$ and $\cos\phi_2 = -\cos\phi$. Equation (3.12) then gives:

$$\omega_1 = \frac{\beta\ c\ \sin\phi}{R\ (1-\beta\cos\phi)} \tag{3.13}$$

$$\omega_2 = \frac{\beta\ c\ \sin\phi}{R\ (1+\beta\cos\phi)} \; . \tag{3.14}$$

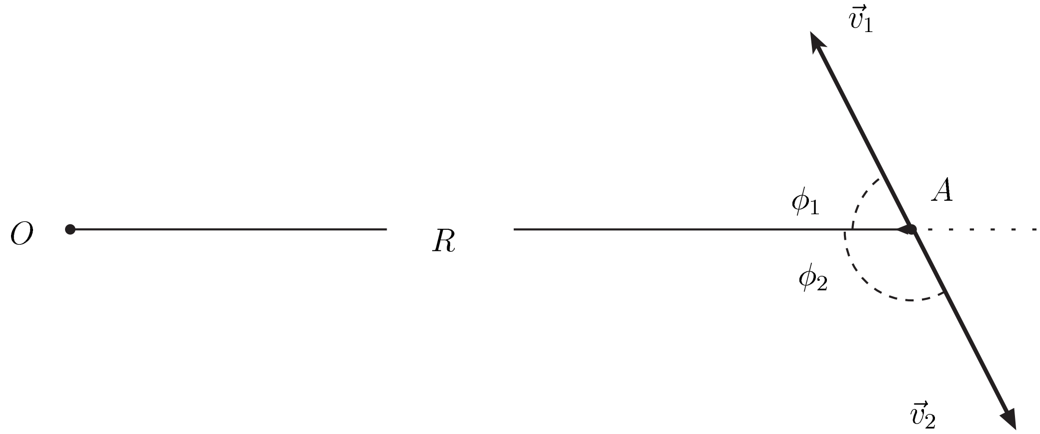


Figure 3.5: *If the two objects have equal speeds but opposite velocities we have* $v_1 = v_2 = v$, $\beta_1 = \beta_2 = \beta$ *and* $\phi_2 = \pi - \phi_1$.

The quantities ω_1, ω_2 and R are given, but β and ϕ are to be determined as stated in the problem text. Simple algebra gives:

$$(1 - \beta \ \cos\phi) \ \omega_1 \ \omega_2 = \beta \ c \ \sin\phi \ \omega_2/R \tag{3.15}$$

$$(1 + \beta \ \cos\phi) \ \omega_2 \ \omega_1 = \beta \ c \ \sin\phi \ \omega_1/R \ . \tag{3.16}$$

Subtracting (3.15) from (3.16) gives:

$$2 \ \beta \ \cos\phi \ \omega_2 \ \omega_1 = \beta \ c \ \sin\phi \ (\omega_1 - \omega_2)/R \tag{3.17}$$

$$\tan\phi = \frac{2 \ R \ \omega_2 \ \omega_1}{c \ \ (\omega_1 - \omega_2)} \tag{3.18}$$

$$\boldsymbol{\phi = \arctan\left(\frac{2 \ R \ \omega_2 \ \omega_1}{c \ \ (\omega_1 - \omega_2)}\right)} \ . \tag{3.19}$$

Dividing (3.15) by (3.16) gives β in terms of $\cos\phi$ and the known quantities ω_1 and ω_2:

$$\omega_1 \ (1 - \beta \ \cos\phi) = \omega_2 \ (1 + \beta \ \cos\phi) \tag{3.20}$$

$$\boldsymbol{\beta = \frac{\omega_1 - \omega_2}{\cos\phi \ (\omega_1 + \omega_2)}} \ . \tag{3.21}$$

Inserting the values of ω_1 and ω_2 from part (a) and the given values of R and c we get:

$$\boldsymbol{\phi} = \arctan(2.57) = \mathbf{1.20 \ rad = 68.8^\circ \pm 2^\circ} \tag{3.22}$$

$$\boldsymbol{\beta} = \mathbf{0.892 \pm 0.08} \tag{3.23}$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$\frac{\beta \sin \phi}{1 - \beta \cos \phi} \geq 1. \tag{3.24}$$

If $\beta < 1$ condition (3.24) is equivalent to:

$$\begin{aligned} \beta \sin \phi &\geq 1 - \beta \cos \phi && (3.25) \\ \beta (\sin \phi + \cos \phi) &\geq 1 && (3.26) \\ \beta \sqrt{2} \left(\sin \phi \cos \frac{\pi}{4} + \cos \phi \sin \frac{\pi}{4} \right) &\geq 1 && (3.27) \\ \sin \left(\phi + \frac{\pi}{4} \right) &\geq \frac{1}{\beta \sqrt{2}} && (3.28) \end{aligned}$$

and hence (3.24) is satisfied if:

$$\beta > \boldsymbol{f(\phi)} = \left(\boldsymbol{\sqrt{2} \sin(\phi + \pi/4)} \right)^{\boldsymbol{-1}} . \tag{3.29}$$

The physically relevant region in the (β, ϕ)-plane is:

$$\boldsymbol{(\beta, \phi) \in [0, 1[\times [0, \pi]} . \tag{3.30}$$

It is obvious that (3.24) can only be satisfied for $\phi \in [0, \pi/2]$ and (3.28) can only have a solution for ϕ if $\beta \geq 1/\sqrt{2}$.

We therefore take a closer look at the region

$$(\beta, \phi) \in [2^{-1/2}, 1[\times [0, \pi/2] \tag{3.31}$$

The mapping

$$(\beta, \phi) \mapsto \beta \sin \left(\phi + \frac{\pi}{4} \right) \tag{3.32}$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

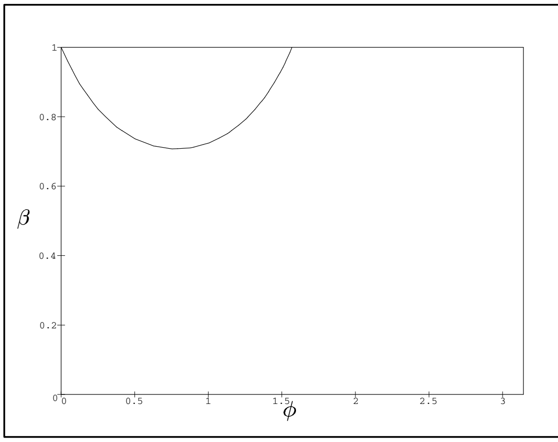
$$\beta \sin \left(\phi + \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} \tag{3.33}$$

This defines β as a function of ϕ which is shown in Figure 3.6 as the curve bounding the shaded area where $v'_\perp > c$.

e) To find the extrema of $v'_\perp$ as a function of ϕ we differentiate (3.11) and get

$$\frac{d}{d\phi} \left(\frac{v'_\perp}{c} \right) = \frac{\beta(\cos \phi - \beta)}{(1 - \beta \cos \phi)^2} . \tag{3.34}$$

This is zero for $\phi = \phi_m$ where:



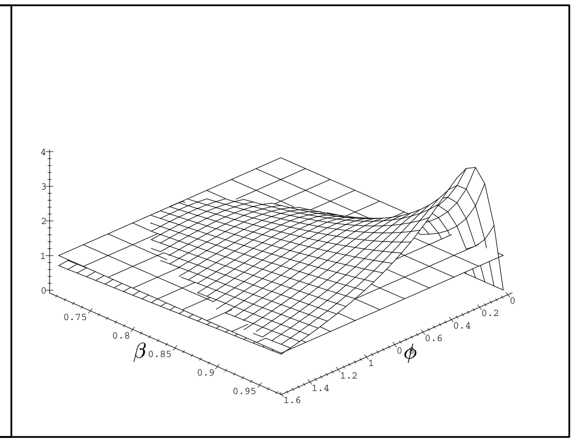


Figure 3.6: *The region between the horizontal line and the curve in the upper left hand corner shows where* $v'_\perp/c > 1$.

Figure 3.7: *The curved surface is* $v'_\perp/c$ *as a function of* β *and* ϕ. *The plane represents the constant function* $\beta = 1$.

$$\cos\phi_m = \beta \;;\; \phi_m = \arccos\beta \in]0, \pi/2] \tag{3.35}$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right) = -\beta\left(\frac{\sin\phi}{(1-\beta\cos\phi)^2} + 2\frac{\beta\sin\phi(\cos\phi-\beta)}{(1-\beta\cos\phi)^3}\right) \tag{3.36}$$

At the extremum

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right)\bigg|_{\phi_m} = -\frac{\beta\sin\phi_m}{(1-\beta^2)^2} < 0 \tag{3.37}$$

showing that ϕ_m corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$\boldsymbol{(v'_\perp)_{max} = \frac{\beta c}{\sqrt{1-\beta^2}}} \tag{3.38}$$

From this and (3.35) we see that

$$(v'_\perp)_{max} \underset{\beta\to1}{\longrightarrow} \infty \;;\; \phi_m \underset{\beta\to1}{\longrightarrow} 0\ . \tag{3.39}$$

Figure 3.7 shows $v'_\perp/c$ as a function of β and ϕ in the region $(\beta, \phi) \in [2^{-1/2}, 1[\times [0, \pi/2]$.

f) We have the equations for relativistic Doppler-shift:

$$\frac{\lambda_{1,2}}{\lambda_0} = \frac{1 \mp \beta \cos\phi}{\sqrt{1-\beta^2}} \tag{3.40}$$

We add them, define an auxiliary ratio ρ and solve for β.

$$\rho := \frac{\lambda_1 + \lambda_2}{2\,\lambda_0} = \frac{1}{\sqrt{1-\beta^2}} \tag{3.41}$$

$$\rho^2\,(1-\beta^2) = 1 \tag{3.42}$$

$$\beta = \sqrt{1 - 1/\rho^2} = \sqrt{1 - \frac{4\,\lambda_0^2}{(\lambda_1+\lambda_2)^2}} \tag{3.43}$$

giving

$$\boldsymbol{\alpha = 4} \tag{3.44}$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns β, ϕ and R. For instance, we may calculate β from (3.43), insert that into (3.21), and solve for ϕ. The distance R can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that ω_1 and ω_2 are known.

3.3 Grading scheme

Part 1(a)	
Answer i): equation (3.2), ω_1 in the range (16.5-17.5) mas/day	**0.8**
Answer ii): equation (3.3), ω_2 in the range (8.2-9.2) mas/day	**0.8**
Answer iii): equation (3.4), for $v'_{1,\perp}$ in the range (1.13-1.30)c	**0.2**
Answer iv): equation (3.6), for $v'_{2,\perp}$ in the range (0.56-0.70)c	**0.2**

Part 1(b)	
Answer i): $v'_\perp(\beta,\phi)$, equation (3.11)	**2.5**
Answer ii): $\omega(\beta,\phi)$, equation (3.12)	**0.5**

Part 1(c)	
Answer i): $\phi(\omega_1,\omega_2)$, equation (3.19)	**0.3**
Answer ii): $\beta(\omega_1,\omega_2)$, equation (3.21)	**0.3**
Answer iii): ϕ numerical in the range 67° - 71°	**0.2**
Answer iv): β numerical in the range 0.81-0.97	**0.2**

Part 1(d)	
Answer i): Condition $\beta > f(\phi)$, equation (3.29)	**1.0**
Answer ii): Condition on (β,ϕ), graph	**1.0**

Part 1(e)	
Answer: $(v'_\perp)_{max}$, equation (3.38)	**1.0**

Part 1(f)	
Answer: β in terms of λ-s, by α, equation (3.44)	**1.0**