Th 1 AN ILL FATED SATELLITE

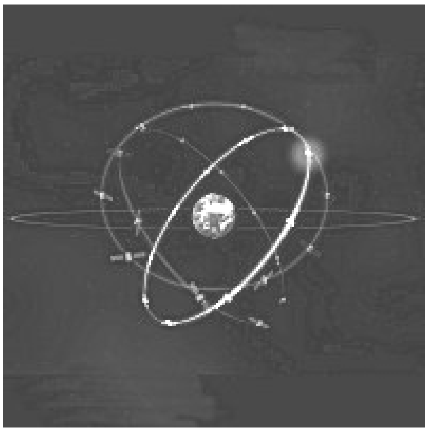


Image: ESA

The most frequent orbital manoeuvres performed by spacecraft consist of velocity variations along the direction of flight, namely accelerations to reach higher orbits or brakings done to initiate re-entering in the atmosphere. In this problem we will study the orbital variations when the engine thrust is applied in a radial direction.

To obtain numerical values use: Earth radius $R_T = 6.37 \cdot 10^6$ m, Earth surface gravity $g = 9.81\,\mathrm{m/s^2}$, and take the length of the sidereal day to be $T_0 = 24.0$ h.

We consider a geosynchronous[1] communications satellite of mass m placed in an equatorial circular orbit of radius r_0. These satellites have an "apogee engine" which provides the tangential thrusts needed to reach the final orbit.

Marks are indicated at the beginning of each subquestion, in parenthesis.

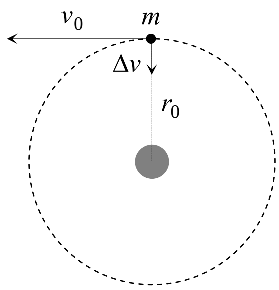
Question 1

1.1 (0.3) Compute the numerical value of r_0.

1.2 (0.3+0.1) Give the analytical expression of the velocity v_0 of the satellite as a function of g, R_T, and r_0, and calculate its numerical value.

1.3 (0.4+0.4) Obtain the expressions of its angular momentum L_0 and mechanical energy E_0, as functions of v_0, m, g and R_T.

Once this geosynchronous circular orbit has been reached (see Figure F-1), the satellite has been stabilised in the desired location, and is being readied to do its work, an error by the ground controllers causes the apogee engine to be fired again. The thrust happens to be directed towards the Earth and, despite the quick reaction of the ground crew to shut the engine off, an unwanted velocity variation Δv is imparted on the satellite. We characterize this boost by the parameter $\beta = \Delta v / v_0$. The duration of the engine burn is always negligible with respect to any other orbital times, so that it can be considered as instantaneous.



F-1

Question 2

Suppose $\beta < 1$.

2.1 (0.4+0.5) Determine the parameters of the new orbit[2], s*emi-latus-rectum* l and *eccentricity* ε, in terms of r_0 and β.

2.2 (1.0) Calculate the angle α between the major axis of the new orbit and the position vector at the accidental misfire.

2.3 (1.0+0.2) Give the analytical expressions of the perigee r_{min} and apogee r_{max} distances to the Earth centre, as functions of r_0 and β, and calculate their numerical values for $\beta = 1/4$.

2.4 (0.5+0.2) Determine the period of the new orbit, T, as a function of T_0 and β, and calculate its numerical value for $\beta = 1/4$.

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[1] Its revolution period is T_0.

[2] See the "hint".

Question 3

3.1 (0.5) Calculate the minimum boost parameter, β_{esc}, needed for the satellite to escape Earth gravity.

3.2 (1.0) Determine in this case the closest approach of the satellite to the Earth centre in the new trajectory, r'_{min}, as a function of r_0.

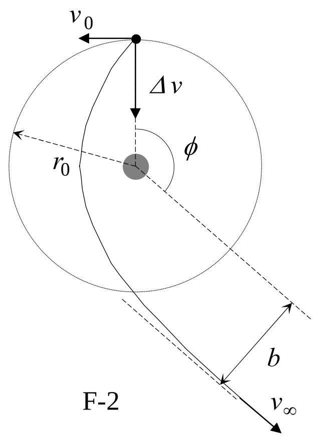
Question 4

Suppose $\beta > \beta_{esc}$.

4.1 (1.0) Determine the residual velocity at the infinity, v_∞, as a function of v_0 and β.

4.2 (1.0) Obtain the "impact parameter" b of the asymptotic escape direction in terms of r_0 and β. (See Figure F-2).

4.3 (1.0+0.2) Determine the angle ϕ of the asymptotic escape direction in terms of β. Calculate its numerical value for $\beta = \frac{3}{2}\beta_{esc}$.



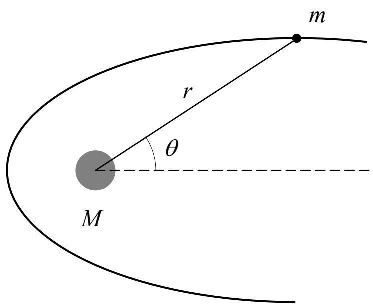
F-2

HINT

Under the action of central forces obeying the inverse-square law, bodies follow trajectories described by ellipses, parabolas or hyperbolas. In the approximation $m << M$ the gravitating mass M is at one of the focuses. Taking the origin at this focus, the general polar equation of these curves can be written as (see Figure F-3)

$$r(\theta) = \frac{l}{1 - \varepsilon\ \cos\theta}$$

where l is a positive constant named the *semi-latus-rectum* and ε is the *eccentricity* of the curve. In terms of constants of motion:



F-3

$$l = \frac{L^2}{G M m^2} \qquad \text{and} \qquad \varepsilon = \left(1 + \frac{2 E L^2}{G^2 M^2 m^3}\right)^{1/2}$$

where G is the Newton constant, L is the modulus of the angular momentum of the orbiting mass, with respect to the origin, and E is its mechanical energy, with zero potential energy at infinity.

We may have the following cases:

i) If $0 \le \varepsilon < 1$, the curve is an ellipse (circumference for $\varepsilon = 0$).

ii) If $\varepsilon = 1$, the curve is a parabola.

iii) If $\varepsilon > 1$, the curve is a hyperbola.

R.S.E.F.

COUNTRY CODE	STUDENT CODE	PAGE NUMBER	TOTAL No OF PAGES

Th 1 ANSWER SHEET

Question	Basic formulas and ideas used	Analytical results	Numerical results	Marking guideline
1.1			$r_0 =$	0.3
1.2		$v_0 =$	$v_0 =$	0.4
1.3		$L_0 =$ $E_0 =$		0.4 0.4
2.1		$l =$ $\varepsilon =$		0.4 0.5
2.2			$\alpha =$	1.0
2.3		$r_{max} =$ $r_{min} =$	$r_{max} =$ $r_{min} =$	1.2
2.4		$T =$	$T =$	0.7
3.1			$\beta_{esc} =$	0.5
3.2		$r'_{min} =$		1.0
4.1		$v_\infty =$		1.0
4.2		$b =$		1.0
4.3		$\phi =$	$\phi =$	1.2

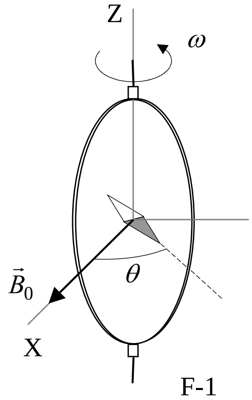
Th 2 ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

The technological and scientific transformations underwent during the XIX century produced a compelling need of universally accepted standards for the electrical quantities. It was thought the new absolute units should only rely on the standards of length, mass and time established after the French Revolution. An intensive experimental work to settle the values of these units was developed from 1861 until 1912. We propose here three case studies.

Marks are indicated at the beginning of each subquestion, in parenthesis.

Determination of the ohm (Kelvin)

A closed circular coil of N turns, radius a and total resistance R is rotated with uniform angular velocity ω about a vertical diameter in a horizontal magnetic field $\vec{B}_0 = B_0\,\vec{i}$.



F-1

1. (0.5+1.0) Compute the electromotive force ε induced in the coil, and also the mean power[1] $\langle P \rangle$ required for maintaining the coil in motion. Neglect the coil self inductance.

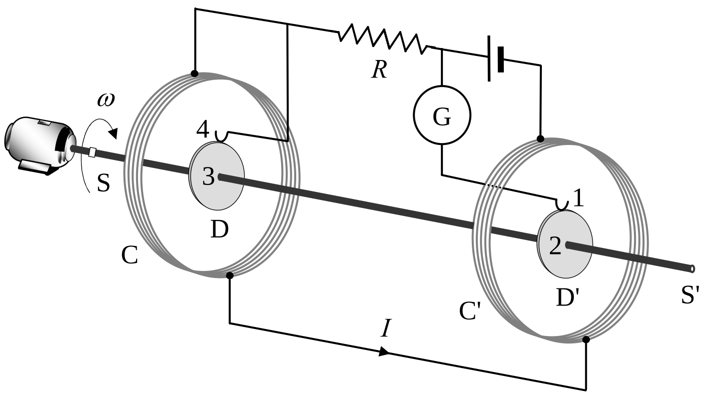
A small magnetic needle is placed at the center of the coil, as shown in Figure F-1. It is free to turn slowly around the Z axis in a horizontal plane, but it cannot follow the rapid rotation of the coil.

2. (2.0) Once the stationary regime is reached, the needle will set at a direction making a small angle θ with $\vec{B}_0$. Compute the resistance R of the coil in terms of this angle and the other parameters of the system.

Lord Kelvin used this method in the 1860s to set the absolute standard for the ohm. To avoid the rotating coil, Lorenz devised an alternative method used by Lord Rayleigh and Ms. Sidgwick, that we analyze in the next paragraphs.

Determination of the ohm (Rayleigh, Sidgwick).

The experimental setup is shown in Figure F-2. It consists of two identical metal disks D and D' of radius b mounted on the conducting shaft SS'. A motor rotates the set at an angular velocity ω, which can be adjusted for measuring R. Two identical coils C and C' (of radius a and with $\boldsymbol{N}$ turns each) surround the disks. They are connected in such a form that the current I flows through them in opposite directions. The whole apparatus serves to measure the resistance R.



F-2

[1] The mean value $\langle X \rangle$ of a quantity $X(t)$ in a periodic system of period T is $\langle X \rangle = \frac{1}{T}\int_0^T X(t)\,dt$

You may need one or more of these integrals:

$$\int_0^{2\pi} \sin x\,dx = \int_0^{2\pi} \cos x\,dx = \int_0^{2\pi} \sin x \cos x\,dx = 0\,, \quad \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = \pi\,, \quad \text{and later} \quad \int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

3. (2.0) Assume that the current I flowing through the coils C and C' creates a uniform magnetic field B around D and D', equal to the one at the centre of the coil. Compute[1] the electromotive force $\mathcal{E}$ induced between the rims 1 and 4, assuming that the distance between the coils is much larger than the radius of the coils and that $a >> b$.

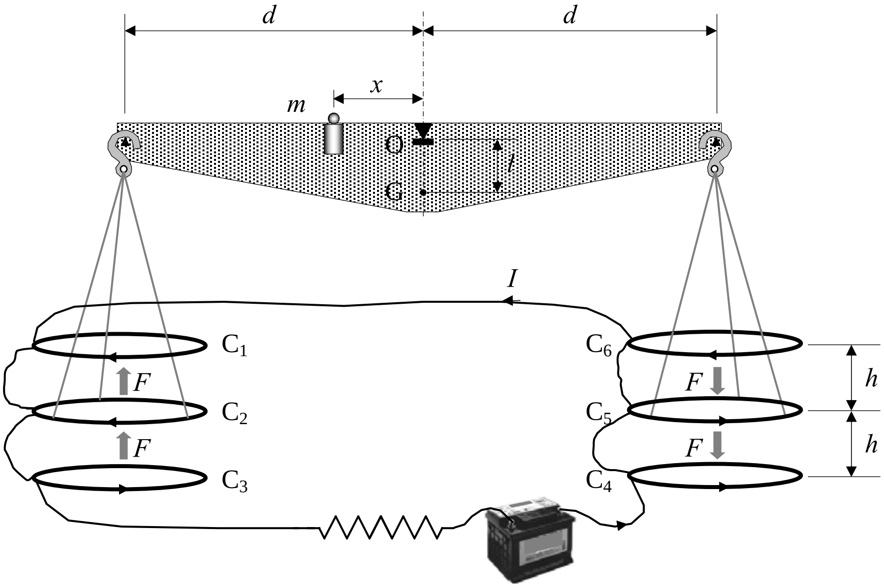
The disks are connected to the circuit by brush contacts at their rims 1 and 4. The galvanometer G detects the flow of current through the circuit 1-2-3-4.

4. (0.5) The resistance R is measured when G reads zero. Give R in terms of the physical parameters of the system.

Determination of the ampere

Passing a current through two conductors and measuring the force between them provides an absolute determination of the current itself. The "Current Balance" designed by Lord Kelvin in 1882 exploits this method. It consists of six identical single turn coils $C_1 \ldots C_6$ of radius a, connected in series. As shown in Figure F-3, the fixed coils C_1, C_3, C_4, and C_6 are on two horizontal planes separated by a small distance $2h$. The coils C_2 and C_5 are carried on balance arms of length d, and they are, in equilibrium, equidistant from both planes.

The current I flows through the various coils in such a direction that the magnetic force on C_2 is upwards while that on C_5 is downwards. A mass m at a distance x from the fulcrum O is required to restore the balance to the equilibrium position described above when the current flows through the circuit.



F-3

5. (1.0) Compute the force F on C_2 due to the magnetic interaction with C_1. For simplicity assume that the force per unit length is the one corresponding to two long, straight wires carrying parallel currents.

6. (1.0) The current I is measured when the balance is in equilibrium. Give the value of I in terms of the physical parameters of the system. The dimensions of the apparatus are such that we can neglect the mutual effects of the coils on the left and on the right.

Let M be the mass of the balance (except for m and the hanging parts), G its centre of mass and l the distance $\overline{OG}$.

7. (2.0) The balance equilibrium is stable against deviations producing small changes δz in the height of C_2 and $-\delta z$ in C_5. Compute[2] the maximum value $\delta z_{\max}$ so that the balance still returns towards the equilibrium position when it is released.

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[2] Consider that the coils centres remain approximately aligned.

Use the approximations $\dfrac{1}{1 \pm \beta} \approx 1 \mp \beta + \beta^2$ or $\dfrac{1}{1 \pm \beta^2} \approx 1 \mp \beta^2$ for $\beta << 1$, and $\sin\theta \approx \tan\theta$ for small θ.

COUNTRY CODE	STUDENT CODE	PAGE NUMBER	TOTAL No OF PAGES

Th 2 ANSWER SHEET

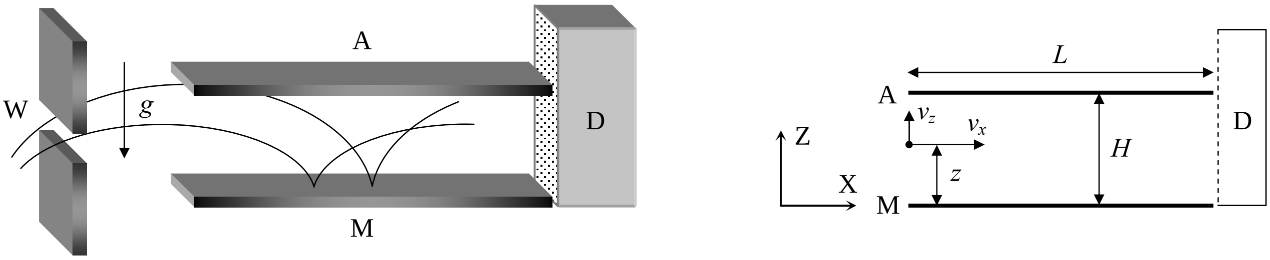
Question	Basic formulas used	Analytical results	Marking guideline
1		$\varepsilon =$ $\langle P \rangle =$	1.5
2		$R =$	2.0
3		$\varepsilon =$	2.0
4		$R =$	0,5
5		$F =$	1.0
6		$I =$	1.0
7		$\delta z_{\max} =$	2.0

Th 3 NEUTRONS IN A GRAVITATIONAL FIELD

In the familiar classical world, an elastic bouncing ball on the Earth's surface is an ideal example for perpetual motion. The ball is trapped: it can not go below the surface or above its turning point. It will remain bounded in this state, turning down and bouncing up once and again, forever. Only air drag or inelastic bounces could stop the process and will be ignored in the following.

A group of physicists from the Institute Laue - Langevin in Grenoble reported[1] in 2002 experimental evidence on the behaviour of neutrons in the gravitational field of the Earth. In the experiment, neutrons moving to the right were allowed to fall towards a horizontal crystal surface acting as a neutron mirror, where they bounced back elastically up to the initial height once and again.

The setup of the experiment is sketched in Figure F-1. It consists of the opening W, the neutron mirror M (at height z = 0), the neutron absorber A (at height $z = H$ and with length L) and the neutron detector D. The beam of neutrons flies with constant horizontal velocity component v_x from W to D through the cavity between A and M. All the neutrons that reach the surface of A are absorbed and disappear from the experiment. Those that reach the surface of M are reflected elastically. The detector D counts the transmission rate $N(H)$, that is, the total number of neutrons that reach D per unit time.



F-1

Marks are indicated at the beginning of each subquestion, in parenthesis.

The neutrons enter the cavity with a wide range of positive and negative vertical velocities, v_z. Once in the cavity, they fly between the mirror below and the absorber above.

1. (1.5) Compute classically the range of vertical velocities $v_z(z)$ of the neutrons that, entering at a height z, can arrive at the detector D. Assume that L is much larger than any other length in the problem.

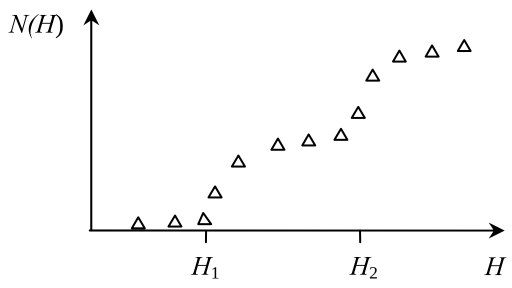
2. (1.5) Calculate classically the minimum length L_c of the cavity to ensure that all neutrons outside the previous velocity range, regardless of the values of z, are absorbed by A. Use $v_x = 10\ \mathrm{m\ s^{-1}}$ and $H = 50\ \mu\mathrm{m}$.

The neutron transmission rate $N(H)$ is measured at D. We expect that it increases monotonically with H.

3. (2.5) Compute the classical rate $N_c(H)$ assuming that neutrons arrive at the cavity with vertical velocity v_z and at height z, being all the values of v_z and z equally probable. Give the answer in terms of ρ, the constant number of neutrons per unit time, per unit vertical velocity, per unit height, that enter the cavity with vertical velocity v_z and at height z.

[1] V. V. Nesvizhevsky et al. "Quantum states of neutrons in the Earth's gravitational field." Nature, **415** (2002) 297. Phys Rev D 67, 102002 (2003).

The experimental results obtained by the Grenoble group disagree with the above classical predictions, showing instead that the value of $N(H)$ experiences sharp increases when H crosses some critical heights H_1, H_2 ... (Figure F-2 shows a sketch). In other words, the experiment showed that the vertical motion of neutrons bouncing on the mirror is quantized. In the language that Bohr and Sommerfeld used to obtain the energy levels of the hydrogen atom, this can be written as: "The action S of these neutrons along the vertical direction is an integer multiple of the Planck action constant h". Here S is given by



F-2

$$S = \int p_z(z)\,dz = n\,h, \quad n = 1, 2, 3\,... \qquad \text{(Bohr-Sommerfeld quantization rule)}$$

where p_z is the vertical component of the classical momentum, and the integral covers a whole bouncing cycle. Only neutrons with these values of S are allowed in the cavity.

4. (2.5) Compute the turning heights H_n and energy levels E_n (associated to the vertical motion) using the Bohr-Sommerfeld quantization condition. Give the numerical result for H_1 in μm and for E_1 in eV.

The uniform initial distribution ρ of neutrons at the entrance changes, during the flight through a long cavity, into the step-like distribution detected at D (see Figure F-2). From now on, we consider for simplicity the case of a long cavity with $H < H_2$. Classically, all neutrons with energies in the range considered in question 1 were allowed through it, while quantum mechanically only neutrons in the energy level E_1 are permitted. According to the time-energy Heisenberg uncertainty principle, this reshuffling requires a minimum time of flight. The uncertainty of the vertical motion energy will be significant if the cavity length is small. This phenomenon will give rise to the widening of the energy levels.

5. (2.0) Estimate the minimum time of flight t_q and the minimum length L_q of the cavity needed to observe the first sharp increase in the number of neutrons at D. Use $v_x = 10\ \text{m s}^{-1}$.

Data:

Planck action constant	$h = 6.63 \cdot 10^{-34}$ J s
Speed of light in vacuum	$c = 3.00 \cdot 10^{8}$ m s^{-1}
Elementary charge	$e = 1.60 \cdot 10^{-19}$ C
Neutron mass	$M = 1.67 \cdot 10^{-27}$ kg
Acceleration of gravity on Earth	$g = 9.81$ m s^{-2}
If necessary, use the expression:	$\int (1-x)^{1/2}\,dx = -\dfrac{2(1-x)^{3/2}}{3}$

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Th 3 ANSWER SHEET

Question	Basic formulas used	Analytical results	Numerical results	Marking guideline
1		$\leq v_z(z) \leq$		1.5
2		$L_c =$	$L_c =$	1.5
3		$N_c(H)=$		2.5
4		$H_n =$ $E_n =$	$H_1=$ μm $E_1 =$ eV	2.5
5		$t_q =$ $L_q =$	$t_q =$ $L_q =$	2.0