Th1 AN ILL FATED SATELLITE
SOLUTION

1.1 and 1.2

$$\left.\begin{array}{l} G\dfrac{M_T m}{r_0^2} = m\dfrac{v_0^2}{r_0} \\ v_0 = \dfrac{2\pi r_0}{T_0} \\ g = \dfrac{GM_T}{R_T^2} \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} r_0 = \left(\dfrac{g\, R_T^2\, T_0^2}{4\pi^2}\right)^{1/3} & \Rightarrow \boxed{r_0 = 4.22\cdot 10^7 \text{ m}} \\ \boxed{v_0 = R_T\sqrt{\dfrac{g}{r_0}}} & \Rightarrow \boxed{v_0 = 3.07\cdot 10^3 \text{ m/s}} \end{array}\right.$$

1.3

$$L_0 = r_0\, m v_0 = \frac{g\, R_T^2}{v_0^2} m v_0 \quad \Rightarrow \quad \boxed{L_0 = \frac{m\, g\, R_T^2}{v_0}}$$

$$E_0 = \frac{1}{2}mv_0^2 - G\frac{M_T m}{r_0} = \frac{1}{2}mv_0^2 - \frac{g\, R_T^2\, m}{r_0} = \frac{1}{2}mv_0^2 - mv_0^2 \quad \Rightarrow \quad \boxed{E_0 = -\frac{1}{2}mv_0^2}$$

2.1

The value of the *semi-latus-rectum* l is obtained taking into account that the orbital angular momentum is the same in both orbits. That is

$$l = \frac{L_0^2}{G M_T\, m^2} = \frac{m^2 g^2 R_T^4}{v_0^2}\frac{1}{g\, R_T^2\, m^2} = \frac{g\, R_T^2}{v_0^2} = r_0 \qquad \Rightarrow \qquad \boxed{l = r_0}$$

The eccentricity value is

$$\varepsilon^2 = 1 + \frac{2\, E\, L_0^2}{G^2 M_T^2\, m^3}$$

where E is the new satellite mechanical energy

$$E = \frac{1}{2}m\left(v_0^2 + \Delta v^2\right) - G\frac{M_T m}{r_0} = \frac{1}{2}m\Delta v^2 + E_0 = \frac{1}{2}m\Delta v^2 - \frac{1}{2}mv_0^2$$

that is

$$E = \frac{1}{2}mv_0^2\left(\frac{\Delta v^2}{v_0^2} - 1\right) = \frac{1}{2}mv_0^2\left(\beta^2 - 1\right)$$

Combining both, one gets $\boxed{\varepsilon = \beta}$

This is an elliptical trajectory because $\varepsilon = \beta < 1$.

2.2

The initial and final orbits cross at P, where the satellite engine fired instantaneously (see Figure 4). At this point

$$r\left(\theta=\alpha\right)=r_0=\frac{r_0}{1-\beta\cos\alpha}\quad\Rightarrow\quad\boxed{\alpha=\frac{\pi}{2}}$$

2.3

From the trajectory expression one immediately obtains that the maximum and minimum values of r correspond to $\theta=0$ and $\theta=\pi$ respectively (see Figure 4). Hence, they are given by

$$r_{max}=\frac{l}{1-\varepsilon}\qquad r_{min}=\frac{l}{1+\varepsilon}$$

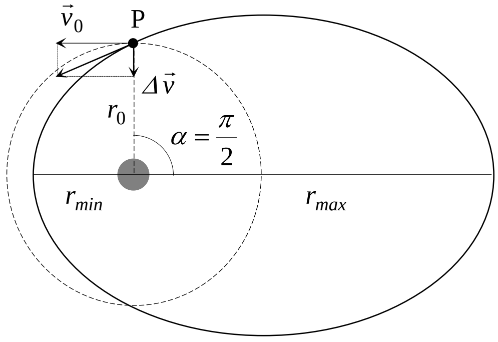


Figure 4

that is

$$\boxed{r_{\max}=\frac{r_0}{1-\beta}}\quad\text{and}\quad\boxed{r_{\min}=\frac{r_0}{1+\beta}}$$

For $\beta=1/4$, one gets

$$\boxed{r_{max}=5.63\cdot10^7\ \text{m};\quad r_{min}=3.38\cdot10^7\ \text{m}}$$

The distances r_{max} and r_{min} can also be obtained from mechanical energy and angular momentum conservation, taking into account that $\vec{r}$ and $\vec{v}$ are orthogonal at apogee and at perigee

$$E=\frac{1}{2}mv_0^2\left(\beta^2-1\right)=\frac{1}{2}mv^2-\frac{gR_T^2m}{r}$$

$$L_0=\frac{mgR_T^2}{v_0}=m\,v\,r$$

What remains of them, after eliminating v, is a second-degree equation whose solutions are r_{max} and r_{min}.

2.4

By the Third Kepler Law, the period T in the new orbit satisfies that

$$\frac{T^2}{a^3}=\frac{T_0^2}{r_0^3}$$

where a, the semi-major axis of the ellipse, is given by

$$a=\frac{r_{max}+r_{min}}{2}=\frac{r_0}{1-\beta^2}$$

Therefore

$$\boxed{T=T_0\left(1-\beta^2\right)^{-3/2}}$$

For $\beta=1/4$ $\qquad\boxed{T=T_0\left(\frac{15}{16}\right)^{-3/2}=26.4\,\text{h}}$

3.1

Only if the satellite follows an open trajectory it can escape from the Earth gravity attraction. Then, the orbit eccentricity has to be equal or larger than one. The minimum boost corresponds to a parabolic trajectory, with $\varepsilon = 1$

$$\varepsilon = \beta \qquad \Rightarrow \qquad \boxed{\beta_{esc} = 1}$$

This can also be obtained by using that the total satellite energy has to be zero to reach infinity (E_p = 0) without residual velocity (E_k = 0)

$$E = \frac{1}{2} m v_0^2 \left(\beta_{esc}^2 - 1\right) = 0 \quad \Rightarrow \qquad \beta_{esc} = 1$$

This also arises from $T = \infty$ or from $r_{max} = \infty$.

3.2

Due to $\varepsilon = \beta_{esc} = 1$, the polar parabola equation is

$$r = \frac{l}{1 - \cos\theta}$$

where the semi-latus-rectum continues to be $l = r_0$. The minimum Earth - satellite distance corresponds to $\theta = \pi$, where

$$\boxed{r'_{min} = \frac{r_0}{2}}$$

This also arises from energy conservation (for E = 0) and from the equality between the angular momenta (L_0) at the initial point P and at maximum approximation, where $\vec{r}$ and $\vec{v}$ are orthogonal.

4.1

If the satellite escapes to infinity with residual velocity v_∞ , by energy conservation

$$E = \frac{1}{2} m v_0^2 \left(\beta^2 - 1\right) = \frac{1}{2} m v_\infty^2 \qquad \Rightarrow$$

$$\boxed{v_\infty = v_0 \left(\beta^2 - 1\right)^{1/2}}$$

4.2

As $\varepsilon = \beta > \beta_{esc} = 1$ the satellite trajectory will be a hyperbola.

The satellite angular momentum is the same at P than at the point where its residual velocity is v_∞ (Figure 5), thus

$$m\, v_0 r_0 = m\, v_\infty b$$

So

$$b = r_0 \frac{v_0}{v_\infty} \quad \Rightarrow \quad \boxed{b = r_0 \left(\beta^2 - 1\right)^{-1/2}}$$

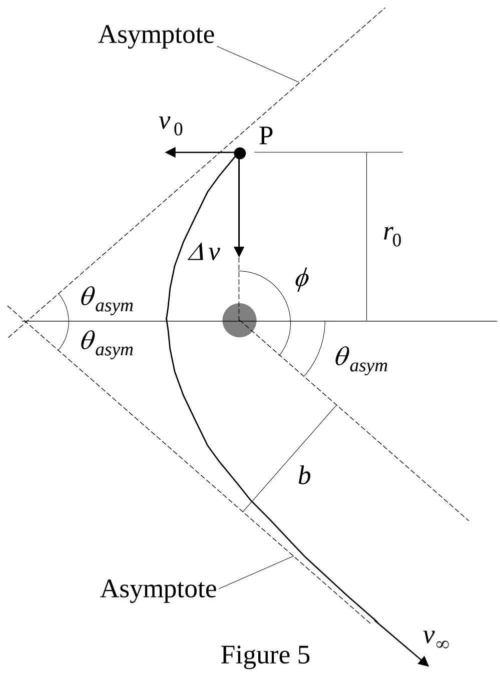


Figure 5

4.3

The angle between each asymptote and the hyperbola axis is that appearing in its polar equation in the limit $r \to \infty$. This is the angle for which the equation denominator vanishes

$$1-\beta\cos\theta_{asym}=0 \quad \Rightarrow \quad \theta_{asym}=\cos^{-1}\left(\frac{1}{\beta}\right)$$

According to Figure 5

$$\phi=\frac{\pi}{2}+\theta_{asym} \quad \Rightarrow \quad \boxed{\phi=\frac{\pi}{2}+\cos^{-1}\left(\frac{1}{\beta}\right)}$$

For $\beta=\frac{3}{2}\beta_{esc}=\frac{3}{2}$, one gets $\boxed{\phi=138^o=2.41\,\mathrm{rad}}$

Th 1 ANSWER SHEET

Question	Basic formulas and ideas used	Analytical results	Numerical results	Marking guideline
1.1	$G\frac{M_T m}{r_0^2} = m\frac{v_0^2}{r_0}$		$r_0 = 4.22\cdot 10^7$ m	0.3
1.2	$v_0 = \frac{2\pi r_0}{T_0}$ $g = \frac{GM_T}{R_T^2}$	$v_0 = R_T\sqrt{\frac{g}{r_0}}$	$v_0 = 3.07\cdot 10^3$ m/s	0.3 + 0.1
1.3	$\vec{L} = m\,\vec{r}\times\vec{v}$ $E = \frac{1}{2}mv^2 - G\frac{Mm}{r}$	$L_0 = \frac{mgR_T^2}{v_0}$ $E_0 = -\frac{1}{2}mv_0^2$		0.4 0.4
2.1	Hint on the conical curves	$l = r_0$ $\varepsilon = \beta$		0.4 0.5
2.2			$\alpha = \frac{\pi}{2}$	1.0
2.3	Results of 2.1, or conservation of E and L	$r_{max} = \frac{r_0}{1-\beta}$ $r_{min} = \frac{r_0}{1+\beta}$	$r_{\max} = 5.63\cdot 10^7$ m $r_{\min} = 3.38\cdot 10^7$ m	1.0 + 0.2
2.4	Third Kepler's Law	$T = T_0\left(1-\beta^2\right)^{-3/2}$	$T = 26.4$ h	0.5 + 0.2
3.1	$\varepsilon = 1$, $E = 0$, $T = \infty$ or $r_{max} = \infty$		$\beta_{esc} = 1$	0.5
3.2	$\varepsilon = 1$ and results of 2.1	$r'_{min} = \frac{r_0}{2}$		1.0
4.1	Conservation of E	$v_\infty = v_0\left(\beta^2 - 1\right)^{1/2}$		1.0
4.2	Conservation of L	$b = r_0\left(\beta^2 - 1\right)^{-1/2}$		1.0
4.3	Hint on the conical curves	$\phi = \frac{\pi}{2} + \cos^{-1}\left(\frac{1}{\beta}\right)$	$\phi = 138^{\circ} = 2.41$ rad	1.0 + 0.2

Th 2 ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

SOLUTION

1. After some time *t*, the normal to the coil plane makes an angle ωt with the magnetic field $\vec{B}_0 = B_0 \vec{i}$. Then, the magnetic flux through the coil is

$$\phi = N \; \vec{B}_0 \cdot \vec{S}$$

where the vector surface $\vec{S}$ is given by $\vec{S} = \pi a^2 \left(\cos \omega t \, \vec{i} + \sin \omega t \, \vec{j} \right)$

Therefore $\phi = N \pi a^2 B_0 \cos \omega t$

The induced electromotive force is

$$\varepsilon = -\frac{d\phi}{dt} \qquad \Rightarrow \qquad \boxed{\varepsilon = N \pi a^2 B_0 \omega \sin \omega t}$$

The instantaneous power is $P = \varepsilon^2 / \mathrm{R}$, therefore

$$\boxed{\langle P \rangle = \frac{\left(N \pi a^2 B_0 \omega \right)^2}{2R}}$$

where we used $< \sin^2 \omega t >= \frac{1}{T} \int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$

2. The total field at the center the coil at the instant *t* is

$$\vec{B}_t = \vec{B}_0 + \vec{B}_i$$

where $\vec{B}_i$ is the magnetic field due to the induced current $\vec{B}_i = B_i \left(\cos \omega t \, \vec{i} + \sin \omega t \, \vec{j} \right)$

with $B_i = \frac{\mu_0 N I}{2a}$ and $I = \varepsilon / R$

Therefore $B_i = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \sin \omega t$

The mean values of its components are

$$\langle B_{ix} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin \omega t \cos \omega t \rangle = 0$$

$$\langle B_{iy} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin^2 \omega t \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{4R}$$

And the mean value of the total magnetic field is

$$\langle \vec{B}_t \rangle = B_0 \, \vec{i} + \frac{\mu_0 N^2 \pi a B_0 \omega}{4R} \vec{j}$$

The needle orients along the mean field, therefore

$$\tan \theta = \frac{\mu_0 N^2 \pi a \omega}{4R}$$

Finally, the resistance of the coil measured by this procedure, in terms of θ, is

$$\boxed{R = \frac{\mu_0 N^2 \pi a \omega}{4 \tan\theta}}$$

3. The force on a unit positive charge in a disk is radial and its modulus is

$$\left|\vec{v} \times \vec{B}\right| = v B = \omega r B$$

where B is the magnetic field at the center of the coil

$$B = N \frac{\mu_0 I}{2a}$$

Then, the electromotive force (e.m.f.) induced on each disk by the magnetic field B is

$$\varepsilon_D = \varepsilon_{D'} = B\omega \int_0^b r\, dr = \frac{1}{2} B \omega b^2$$

Finally, the induced e.m.f. between 1 and 4 is $\varepsilon = \varepsilon_D + \varepsilon_{D'}$

$$\boxed{\varepsilon = N \frac{\mu_0 b^2 \omega I}{2a}}$$

4. When the reading of G vanishes, $I_G = 0$ and Kirchoff laws give an immediate answer. Then we have

$$\varepsilon = I R \qquad \Rightarrow \qquad \boxed{R = N \frac{\mu_0 b^2 \omega}{2a}}$$

5. The force per unit length f between two indefinite parallel straight wires separated by a distance h is.

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{h}$$

for $I_1 = I_2 = I$ and length $2\pi a$, the force F induced on C_2 by the neighbor coils C_1 is

$$\boxed{F = \frac{\mu_0 a}{h} I^2}$$

6. In equilibrium

$$m g x = 4 F d$$

Then

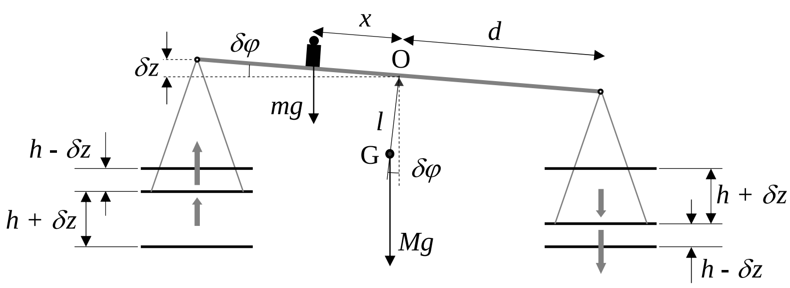
$$m g x = \frac{4 \mu_0 a d}{h} I^2 \qquad (1)$$

so that

$$\boxed{I = \left(\frac{m g h x}{4 \mu_0 a d} \right)^{1/2}}$$

7. The balance comes back towards the equilibrium position for a little angular deviation $\delta\varphi$ if the gravity torques with respect to the fulcrum O are greater than the magnetic torques.

$$Mgl\sin\delta\varphi + mg\,x\cos\delta\varphi > 2\mu_0\,aI^2\left(\frac{1}{h-\delta z}+\frac{1}{h+\delta z}\right)d\cos\delta\varphi$$



Therefore, using the suggested approximation

$$Mgl\sin\delta\varphi + mg\,x\cos\delta\varphi > \frac{4\mu_0\,adI^2}{h}\left(1+\frac{\delta z^2}{h^2}\right)\cos\delta\varphi$$

Taking into account the equilibrium condition (1), one obtains

$$M\,g\,l\,sin\,\delta\varphi > m\,g\,x\frac{\delta z^2}{h^2}cos\,\delta\varphi$$

Finally, for $\tan\delta\varphi \approx \sin\delta\varphi = \dfrac{\delta z}{d}$

$$\delta z < \frac{Mlh^2}{mxd} \qquad \Rightarrow \qquad \boxed{\delta z_{\max} = \frac{Mlh^2}{mxd}}$$

Th 2 ANSWER SHEET

Question	Basic formulas and ideas used	Analytical results	Marking guideline
1	$\Phi = N\,\vec{B}_0 \cdot \vec{S}$ $\varepsilon = -\dfrac{d\Phi}{dt}$ $P = \dfrac{\varepsilon^2}{R}$	$\varepsilon = N\pi a^2 B_0 \omega \sin\omega t$ $\langle P \rangle = \dfrac{\left(N\pi a^2 B_0 \omega\right)^2}{2R}$	0.5 1.0
2	$\vec{B} = \vec{B}_0 + \vec{B}_i$ $B_i = \dfrac{\mu_0 N}{2a} I$ $\tan\theta = \dfrac{\langle B_y \rangle}{\langle B_x \rangle}$	$R = \dfrac{\mu_0 N^2 \pi a \omega}{4\tan\theta}$	2.0
3	$\vec{E} = \vec{v} \times \vec{B}$ $v = \omega r$ $B = N\dfrac{\mu_0 I}{2a}$ $\varepsilon = \int_0^b \vec{E}\, d\vec{r}$	$\varepsilon = N\dfrac{\mu_0 b^2 \omega I}{2a}$	2.0
4	$\varepsilon = R I$	$R = N\dfrac{\mu_0 b^2 \omega}{2a}$	0,5
5	$f = \dfrac{\mu_0}{2\pi}\dfrac{I I'}{h}$	$F = \dfrac{\mu_0\, a}{h} I^2$	1.0
6	$m g x = 4F d$	$I = \left(\dfrac{m g h x}{4\mu_0\, a d}\right)^{1/2}$	1.0
7	$\Gamma_{grav} > \Gamma_{mag}$	$\delta z_{\max} = \dfrac{M l h^2}{m x d}$	2.0

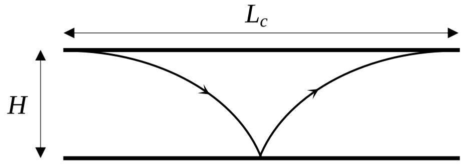
Th3 QUANTUM EFFECTS OF GRAVITY

SOLUTION

1. The only neutrons that will survive absorption at A are those that cannot cross H. Their turning points will be below H. So that, for a neutron entering to the cavity at height z with vertical velocity v_z, conservation of energy implies

$$\frac{1}{2}M\,v_z^2 + M\,g\,z \le M\,g\,H \qquad \Rightarrow \qquad \boxed{-\sqrt{2g(H-z)} \le v_z(z) \le \sqrt{2g(H-z)}}$$

2. The cavity should be long enough to ensure the absorption of all neutrons with velocities outside the allowed range. Therefore, neutrons have to reach its maximum height at least once within the cavity. The longest required length corresponds to neutrons that enter at $z = H$ with $v_z = 0$ (see the figure). Calling t_f to their time of fall



$$\left.\begin{aligned} L_c &= v_x\, 2 t_f \\ H &= \frac{1}{2} g\, t_f^2 \end{aligned}\right\} \quad \Rightarrow \quad \boxed{L_c = 2 v_x \sqrt{\frac{2H}{g}}} \qquad \boxed{L_c = 6.4 \text{ cm}}$$

3. The rate of transmitted neutrons entering at a given height z, per unit height, is proportional to the range of allowed velocities at that height, ρ being the proportionality constant

$$\frac{dN_c(z)}{dz} = \rho\left[v_{z,\max}(z) - v_{z,\min}(z)\right] = 2\rho\sqrt{2g(H-z)}$$

The total number of transmitted neutrons is obtained by adding the neutrons entering at all possible heights. Calling $y = z/H$

$$N_c(H) = \int_0^H dN_c(z) = \int_0^H 2\rho\sqrt{2g(H-z)}dz = 2\rho\sqrt{2g}\,H^{3/2}\int_0^1 (1-y)^{1/2}dy = 2\rho\sqrt{2g}\,H^{3/2}\left[-\frac{2}{3}(1-y)^{3/2}\right]_0^1$$

$$\Rightarrow \qquad \boxed{N_c(H) = \frac{4}{3}\rho\sqrt{2g}\,H^{3/2}}$$

4. For a neutron falling from a height H, the action over a bouncing cycle is twice the action during the fall or the ascent

$$S = 2\int_0^H p_z dz = 2M\sqrt{2g}\,H^{3/2}\int_0^1 (1-y)^{1/2}dy = \frac{4}{3}M\sqrt{2g}\,H^{3/2}$$

Using the BS quantization condition

$$S = \frac{4}{3}M\sqrt{2g}\,H^{3/2} = n\,h \qquad \Rightarrow \qquad \boxed{H_n = \left(\frac{9\,h^2}{32\,M^2 g}\right)^{1/3} n^{2/3}}$$

The corresponding energy levels (associated to the vertical motion) are

$$E_n = M\,g\,H_n \qquad \Rightarrow \qquad \boxed{E_n = \left(\frac{9M\,g^2 h^2}{32}\right)^{1/3} n^{2/3}}$$

Numerical values for the first level:

$$H_1 = \left(\frac{9 h^2}{32 M^2 g} \right)^{1/3} = 1.65 \times 10^{-5} \text{ m} \qquad \boxed{H_1 = 16.5 \ \mu\text{m}}$$

$$E_1 = M g H_1 = 2.71 \times 10^{-31} \text{ J} = 1.69 \times 10^{-12} \text{ eV} \qquad \boxed{E_1 = 1.69 \text{ peV}}$$

Note that H_1 is of the same order than the given cavity height, H = 50 μm. This opens up the possibility for observing the spatial quantization when varying H.

5. The uncertainty principle says that the minimum time Δt and the minimum energy ΔE satisfy the relation $\Delta E \, \Delta t \geq \hbar$. During this time, the neutrons move to the right a distance

$$\Delta x = v_x \Delta t \geq v_x \frac{\hbar}{\Delta E}$$

Now, the minimum neutron energy allowed in the cavity is E_1, so that $\Delta E \approx E_1$. Therefore, an estimation of the minimum time and the minimum length required is

$$\boxed{t_q \approx \frac{\hbar}{E_1} = 0.4 \cdot 10^{-3} \text{ s} = 0.4 \text{ ms}} \qquad \boxed{L_q \approx v_x \frac{\hbar}{E_1} = 4 \cdot 10^{-3} \text{ m} = 4 \text{ mm}}$$

Th 3 ANSWER SHEET

Question	Basic formulas used	Analytical results	Numerical results	Marking guideline
1	$\frac{1}{2}M\,v_z^2 + M\,g\,z \le M\,g\,H$	$-\sqrt{2g(H-z)} \le v_z(z) \le \sqrt{2g(H-z)}$		1.5
2	$L_c = v_x 2t_f$ $H = \frac{1}{2}g t_f^2$	$L_c = 2v_x\sqrt{\frac{2H}{g}}$	$L_c = 6.4$ cm	1.3 + 0.2
3	$\frac{dN_c}{dz} = \rho\left[v_{z,\max} - v_{z,\min}\right]$ $N_c(H) = \int_0^H dN_c(z)$	$N_c(H) = \frac{4}{3}\rho\sqrt{2g}\,H^{3/2}$		2.5
4	$S = 2\int_0^H p_z dz = nh$	$H_n = \left(\frac{9h^2}{32M^2g}\right)^{1/3} n^{2/3}$ $E_n = \left(\frac{9M\,g^2h^2}{32}\right)^{1/3} n^{2/3}$	$H_1 = 16.5$ μm $E_1 = 1.69$ peV	1.6 + 0.2 0.5 + 0.2
5	$\Delta E\,\Delta t \ge \hbar$ $\Delta E \approx E_1$ $\Delta x = v_x \Delta t$	$t_q \approx \frac{\hbar}{E_1}$ $L_q \approx v_x\frac{\hbar}{E_1}$	$t_q \approx 0.4$ ms $L_q \approx 4$ mm	1.3 + 0.2 0.3 + 0.2