1.1) One may use any reasonable equation to obtain the dimension of the questioned quantities.

I) The Planck relation is $h\nu = E \quad \Rightarrow \quad [h][\nu] = [E] \quad \Rightarrow \quad [h] = [E][\nu]^{-1} = ML^2T^{-1}$
(0.2)

II) $[c] = LT^{-1}$ (0.2)

III) $F = \dfrac{Gmm}{r^2} \quad \Rightarrow \quad [G] = [F][r^2][m]^{-2} = M^{-1}L^3T^{-2}$ (0.2)

IV) $E = K_B\theta \quad \Rightarrow \quad [K_B] = [\theta]^{-1}[E] = ML^2T^{-2}K^{-1}$ (0.2)

1.2) Using the Stefan-Boltzmann's law,
$\dfrac{Power}{Area} = \sigma\theta^4$, or any equivalent relation, one obtains:
(0.3)
$[\sigma]K^4 = [E]L^{-2}T^{-1} \Rightarrow [\sigma] = MT^{-3}K^{-4}.$ (0.2)

1.3) The Stefan-Boltzmann's constant, up to a numerical coefficient, equals $\sigma = h^\alpha c^\beta G^\gamma k_B{}^\delta$, where $\alpha, \beta, \gamma, \delta$ can be determined by dimensional analysis. Indeed, $[\sigma] = [h]^\alpha [c]^\beta [G]^\gamma [k_B]^\delta$, where e.g. $[\sigma] = MT^{-3}K^{-4}$.

$$MT^{-3}K^{-4} = \left(ML^2T^{-1}\right)^\alpha \left(LT^{-1}\right)^\beta \left(M^{-1}L^3T^{-2}\right)^\gamma \left(ML^2T^{-2}K^{-1}\right)^\delta = M^{\alpha-\gamma+\delta}L^{2\alpha+\beta+3\gamma+2\delta}T^{-\alpha-\beta-2\gamma-2\delta}K^{-\delta},$$
(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} \alpha - \gamma + \delta = 1, \\ 2\alpha + \beta + 3\gamma + 2\delta = 0, \\ -\alpha - \beta - 2\gamma - 2\delta = -3, \\ -\delta = -4, \end{cases}$$ (Each one (0.1)) $$\Rightarrow \begin{cases} \alpha = -3, \\ \beta = -2, \\ \gamma = 0, \\ \delta = 4. \end{cases}$$ (Each one (0.1))

$$\Rightarrow \qquad \sigma = \frac{k_B{}^4}{c^2h^3}.$$

2.1) Since A, the area of the event horizon, is to be calculated in terms of m from a classical theory of relativistic gravity, e.g. the General Relativity, it is a combination of c, characteristic of special relativity, and G characteristic of gravity. Especially, it is independent of the Planck constant h which is characteristic of quantum mechanical phenomena.

$$A = G^{\alpha} c^{\beta} m^{\gamma}$$

Exploiting dimensional analysis,

$$\Rightarrow \ [A] = [G]^{\alpha}[c]^{\beta}[m]^{\gamma} \ \Rightarrow \ L^2 = \left(M^{-1}L^3T^{-2}\right)^{\alpha}\left(LT^{-1}\right)^{\beta} M^{\gamma} = M^{-\alpha+\gamma}L^{3\alpha+\beta}T^{-2\alpha-\beta}$$

(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} -\alpha + \gamma = 0, \\ 3\alpha + \beta = 2, \\ -2\alpha - \beta = 0, \end{cases} \text{(Each one (0.1))} \quad \Rightarrow \quad \begin{cases} \alpha = 2, \\ \beta = -4, \\ \gamma = 2, \end{cases} \text{(Each one (0.1))} \Rightarrow$$

$$A = \frac{m^2 G^2}{c^4}.$$

2.2)

From the definition of entropy $dS = \dfrac{dQ}{\theta}$, one obtains $[S] = [E][\theta]^{-1} = ML^2T^{-2}K^{-1}$ (0.2)

2.3) Noting $\eta = S/A$, one verifies that,

$$\begin{cases} [\eta] = [S][A]^{-1} = MT^{-2}K^{-1}, \\ [\eta] = [G]^{\alpha}[h]^{\beta}[c]^{\gamma}[k_B]^{\delta} = M^{-\alpha+\beta+\delta}L^{3\alpha+2\beta+\gamma+2\delta}T^{-2\alpha-\beta-\gamma-2\delta}K^{-\delta}, \end{cases}$$

(0.2)

Using the same scheme as above,

$$\Rightarrow \begin{cases} -\alpha + \beta + \delta = 1, \\ 3\alpha + 2\beta + \gamma + 2\delta = 0, \\ -2\alpha - \beta - \gamma - 2\delta = -2, \\ \delta = 1, \end{cases} \text{(Each one (0.1))} \quad \Rightarrow \begin{cases} \alpha = -1, \\ \beta = -1, \\ \gamma = 3, \\ \delta = 1, \end{cases} \text{(Each one (0.1))}$$

thus, $\eta = \dfrac{c^3 k_B}{G h}$. (0.1)

3.1)

The first law of thermodynamics is $dE = dQ + dW$. By assumption, $dW = 0$. Using the definition of entropy, $dS = \frac{dQ}{\theta}$, one obtains,

$$dE = \theta_H \, dS + 0,$$ (0.2) + (0.1), for setting $dW = 0$.

Using, $\begin{cases} S = \dfrac{Gk_B}{ch} m^2, \\ E = mc^2, \end{cases}$ [(0.1) for S]

one obtains, $\theta_H = \dfrac{dE}{dS} = \left(\dfrac{dS}{dE}\right)^{-1} = c^2 \left(\dfrac{dS}{dm}\right)^{-1}$ (0.2)

Therefore, $\theta_H = \left(\dfrac{1}{2}\right) \dfrac{c^3 h}{Gk_B} \dfrac{1}{m}$. (0.1)+(0.1) (for the coefficient)

3.2) The Stefan-Boltzmann's law gives the rate of energy radiation per unit area. Noting that $E = mc^2$ we have:

$$\begin{cases} dE/dt = -\sigma \theta_H{}^4 A, & (0.2) \\ \sigma = \dfrac{k_B{}^4}{c^2 h^3}, \\ A = \dfrac{m^2 G^2}{c^4} \\ E = mc^{2,} \end{cases} \Rightarrow c^2 \frac{dm}{dt} = -\frac{k_B{}^4}{c^2 h^3} \left(\frac{c^3 h}{2G k_B} \frac{1}{m} \right)^4 \frac{m^2 G^2}{c^4}, \quad (0.2)$$

$$\Rightarrow \quad \frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}.$$ (0.1) (for simplification) + (0.2) (for the minus sign)

3.3)
By integration:

$$\frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}. \quad \Rightarrow \int m^2 dm = -\int \frac{c^4 h}{16 G^2} dt$$ (0.3)

$$\Rightarrow m^3(t) - m^3(0) = -\frac{3c^4 h}{16 G^2} t,$$ (0.2) + (0.2) (Integration and correct boundary values)

At $t = t^*$ the black hole evaporates completely:

$m(t^*) = 0$ (0.1) $\Rightarrow t^* = \dfrac{16G^2}{3c^4 h} m^3$ (0.2)+(0.1) (for the coefficient)

3.4) C_V measures the change in E with respect to variation of θ.

$$\begin{cases} C_V = \dfrac{dE}{d\theta}, & \text{(0.2)} \\ E = mc^2, & \text{(0.2)} \\ \theta = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m} \end{cases} \Rightarrow \quad C_V = -\frac{2Gk_B}{ch} m^2.$$

0.1)+(0.1) (for the coefficient)

4.1) Again the Stefan-Boltzmann's law gives the rate of energy loss per unit area of the black hole. A similar relation can be used to obtain the energy gained by the black hole due to the background radiation. To justify it, note that in the thermal equilibrium, the total change in the energy is vanishing. The blackbody radiation is given by the Stefan-Boltzmann's law. Therefore the rate of energy gain is given by the same formula.

(0.1) + (0.4) (For the first and the second terms respectively)

$$\begin{cases} \dfrac{dE}{dt} = -\sigma\theta^4 A + \sigma\theta_B{}^4 A \\ E = mc^2, \end{cases} \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^2$$

(0.3)

4.2)

Setting $\dfrac{dm}{dt} = 0$, we have:

$$-\frac{hc^4}{16G^2}\frac{1}{m^{*2}} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^{*2} = 0$$

(0.2)

and consequently,

$$m^* = \frac{c^3 h}{2G\,k_B}\frac{1}{\theta_B}$$

(0.2)

4.3)

$$\theta_B = \frac{c^3 h}{2G\,k_B}\frac{1}{m^*} \quad \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2}\left(1 - \frac{m^4}{m^{*4}}\right)$$

(0.2)

4.4) Use the solution to 4.2,

$m^* = \dfrac{c^3 h}{2G\,k_B}\dfrac{1}{\theta_B}$ (0.2) and 3.1 to obtain, $\theta^* = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m^*} = \theta_B$ (0.2)

One may also argue that m^* corresponds to thermal equilibrium. Thus for $m = m^*$ the black hole temperature equals θ_B.

Or one may set $\dfrac{dE}{dt} = -\sigma\left(\theta^{*4} - \theta_B{}^4\right)A = 0$ to get $\theta^* = \theta_B$.

4.5) Considering the solution to 4.3, one verifies that it will go away from the equilibrium. (0.6)

$$\frac{dm}{dt} = -\frac{hc^4}{G^2}\frac{1}{m^2}\left(1 - \frac{m^4}{m^{*4}}\right) \Rightarrow \begin{cases} m > m^* & \Rightarrow \quad \frac{dm}{dt} > 0 \\ m < m^* & \Rightarrow \quad \frac{dm}{dt} < 0 \end{cases}$$

Question “Orange”

1.1)
First of all, we use the Gauss’s law for a single plate to obtain the electric field,

$$E = \frac{\sigma}{\varepsilon_0}. \qquad (0.2)$$

The density of surface charge for a plate with charge, Q and area, A is

$$\sigma = \frac{Q}{A}. \qquad (0.2)$$

Note that the electric field is resulted by two equivalent parallel plates. Hence the contribution of each plate to the electric field is $\frac{1}{2}E$. Force is defined by the electric filed times the charge, then we have

Force = $\frac{1}{2}EQ = \frac{Q^2}{2\varepsilon_0 A}$ (0.2)+ (0.2) (The ½ coefficient + the final result)

1.2)
The Hook’s law for a spring is

$F_m = -k\,x$. (0.2)

In 1.2 we derived the electric force between two plates is

$$F_e = \frac{Q^2}{2\varepsilon_0 A}.$$

The system is stable. The equilibrium condition yields

$F_m = F_e$, (0.2)

$$\Rightarrow \qquad x = \frac{Q^2}{2\varepsilon_0 A k} \qquad (0.2)$$

1.3)
The electric field is constant thus the potential difference, V is given by

$V = E(d - x)$ (0.2)

(Other reasonable approaches are acceptable. For example one may use the definition of capacity to obtain V.)

By substituting the electric field obtained from previous section to the above equation, we get,

$$V = \frac{Q\,d}{\varepsilon_0 A}\left(1 - \frac{Q^2}{2\varepsilon_0 A\,k\,d}\right) \qquad (0.2)$$

1.4)
C is defined by the ratio of charge to potential difference, then

$$C = \frac{Q}{V}. \qquad (0.1)$$

Using the answer to 1.3, we get $\dfrac{C}{C_0} = \left(1 - \dfrac{Q^2}{2\varepsilon_0 A k d}\right)^{-1}$ (0.2)

1.5)

Note that we have both the mechanical energy due to the spring

$$U_m = \frac{1}{2}kx^2, \qquad (0.2)$$

and the electrical energy stored in the capacitor.

$$U_E = \frac{Q^2}{2C}. \qquad (0.2)$$

Therefore the total energy stored in the system is

$$U = \frac{Q^2 d}{2\varepsilon_0 A}\left(1 - \frac{Q^2}{4\varepsilon_0 A k d}\right) \qquad (0.2)$$

2.1)

For the given value of x, the amount of charge on each capacitor is

$$Q_1 = V\,C_1 = \frac{\varepsilon_0 A V}{d - x}, \qquad (0.2)$$

$$Q_2 = V\,C_2 = \frac{\varepsilon_0 A V}{d + x}. \qquad (0.2)$$

2.2)

Note that we have two capacitors. By using the answer to 1.1 for each capacitor, we get

$$F_1 = \frac{Q_1^{\,2}}{2\varepsilon_0 A},$$

$$F_2 = \frac{Q_2^{\,2}}{2\varepsilon_0 A}.$$

As these two forces are in the opposite directions, the net electric force is

$$F_E = F_1 - F_2, \quad (0.2) \quad \Rightarrow \quad F_E = \frac{\varepsilon_0 A V^2}{2}\left(\frac{1}{(d-x)^2} - \frac{1}{(d+x)^2}\right) \quad (0.2)$$

2.3)

Ignoring terms of order x^2 in the answer to 2.2., we get

$$F_E = \frac{2\varepsilon_0 A V^2}{d^3}x \qquad (0.2)$$

2.4)

There are two springs placed in series with the same spring constant, k, then the mechanical force is

$F_m = -2kx$. (The coefficient (2) has (0.2))

Combining this result with the answer to 2.4 and noticing that these two forces are in the opposite directions, we get

$F = F_m + F_E$, $\Rightarrow$ $F = -2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x$, (Opposite signs of the two forces have (0.3))

$\Rightarrow \quad k_{eff} = 2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)$ (0.2)

2.5)

By using the Newtown's second law,

$F = ma$ (0.2)

and the answer to 2.4, we get

$a = -\frac{2}{m}\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x$ (0.2)

3.1)

Starting with Kirchhoff's laws, for two electrical circuits, we have

$$\begin{cases} \frac{Q_S}{C_S} + V - \frac{Q_2}{C_2} = 0 \\ -\frac{Q_S}{C_S} + V - \frac{Q_1}{C_1} = 0 \\ Q_2 - Q_1 + Q_S = 0 \end{cases}$$

(Each has (0.3), Note: the sings may depend on the specific choice made)

Noting that $V_s = \frac{Q_S}{C_S}$ one obtains

$$\Rightarrow V_S = V \frac{\frac{2\varepsilon_0 A x}{d^2 - x^2}}{C_S + \frac{2\varepsilon_0 A d}{d^2 - x^2}}.$$

((0.4) + (0.2): (0.4) for solving the above equations and (0.2) for final result)

Note: Students may simplify the above relation using the approximation $d^2 >> x^2$. It does not matter in this section.

3.2)

Ignoring terms of order x^2 in the answer to 3.1., we get

$$V_S = V \frac{2\varepsilon_0 A x}{d^2 C_S + 2\varepsilon_0 A d} \ . \qquad (0.2)$$

4.1)

The ratio of the electrical force to the mechanical (spring) force is

$$\frac{F_E}{F_m} = \frac{\varepsilon_0 A V^2}{k d^3} \ ,$$

Putting the numerical values:

$$\frac{F_E}{F_m} = 7.6\times 10^{-9} \ .$$

((0.2) + (0.2) + (0.2): (0.2) for order of magnitude, (0.2) for two significant digits and (0.2) for correct answer (7.6 or 7.5)).

As it is clear from this result, we can ignore the electrical forces compared to the electric force.

4.2)

As seen in the previous section, one may assume that the only force acting on the moving plate is due to springs:

$F = 2k x$. (The concept of equilibrium (0.2))

Hence in mechanical equilibrium, the displacement of the moving plate is

$$x = \frac{ma}{2k} \ .$$

The maximum displacement is twice this amount, like the mass spring system in a gravitational force field, when the mass is let to fall.

$x_{\max} = 2x$ (0.2)

$x_{\max} = \dfrac{ma}{k}$ (0.2)

4.3)

At the acceleration

$a = g$, (0.2)

The maximum displacement is

$$x_{\max} = \frac{mg}{k} \ .$$

Moreover, from the result obtained in 3.2, we have

$$V_S = V\frac{2\varepsilon_0 A\, x_{\max}}{d^2 C_S + 2\varepsilon_0 A\, d}$$

This should be the same value given in the problem, $0.15\,\mathrm{V}$.

$$\Rightarrow \qquad C_S = \frac{2\varepsilon_0 A}{d}\left(\frac{V\, x_{\max}}{V_S\, d} - 1\right) \qquad (0.2)$$

$$\Rightarrow \qquad C_S = 8.0\times 10^{-11}\,\mathrm{F} \qquad (0.2)$$

4.4)

Let ℓ be the distance between the driver's head and the steering wheel. It can be estimated to be about

$\ell = 0.4\,m - 1\,m$. (0.2)

Just at the time the acceleration begins, the relative velocity of the driver's head with respect to the automobile is zero.

$\Delta v(t=0) = 0$, (0.2)

then

$$\ell = \frac{1}{2} g\, t_1^{\,2} \qquad \Rightarrow \qquad t_1 = \sqrt{\frac{2\,\ell}{g}} \qquad (0.2)$$

$t_1 = 0.3 - 0.5\ s$ (0.2)

4.5)

The time t_2 is half of period of the harmonic oscillator, hence

$$t_2 = \frac{T}{2}, \qquad (0.3)$$

The period of harmonic oscillator is simply given by

$$T = 2\pi\sqrt{\frac{m}{2k}}\ , \qquad (0.2)$$

therefore,

$t_2 = 0.013\ s$. (0.2)

As $t_1 > t_2$, the airbag activates in time. (0.2)

Question “Pink”

1.1

Period = 3.0 days = $2.6\times10^{5}\,\text{s}$. (0.4)

Period = $\dfrac{2\pi}{\omega}$ (0.2) $\Rightarrow$ $\omega = 2.4\times10^{-5}$ rad s^{-1}. (0.2)

1.2

Calling the minima in the diagram 1, $I_1/I_0 = \alpha = 0.90$ and $I_2/I_0 = \beta = 0.63$, we have:

$$\frac{I_0}{I_1} = 1 + \left(\frac{R_2}{R_1}\right)^2 \left(\frac{T_2}{T_1}\right)^4 = \frac{1}{\alpha} \quad \text{(0.4)}$$

$$\frac{I_2}{I_1} = 1 - \left(\frac{R_2}{R_1}\right)^2 \left(1 - \left(\frac{T_2}{T_1}\right)^4\right) = \frac{\beta}{\alpha} \quad \text{(0.4)} \quad \text{(or equivalent relations)}$$

From above, one finds:

$$\frac{R_1}{R_2} = \sqrt{\frac{\alpha}{1-\beta}} \Rightarrow \frac{R_1}{R_2} = 1.6 \quad \text{(0.2+0.2)} \quad \text{and} \quad \frac{T_1}{T_2} = \sqrt[4]{\frac{1-\beta}{1-\alpha}} \Rightarrow \frac{T_1}{T_2} = 1.4 \quad \text{(0.2+0.2)}$$

2.1)

Doppler-Shift formula:

$\dfrac{\Delta\lambda}{\lambda_0} \cong \dfrac{v}{c}$ (or equivalent relation) (0.4)

Maximum and minimum wavelengths: $\lambda_{1,\max} = 5897.7$ Å , $\lambda_{1,\min} = 5894.1$ Å

$\lambda_{2,\max} = 5899.0$ Å , $\lambda_{2,\min} = 5892.8$ Å

Difference between maximum and minimum wavelengths:

$\Delta\lambda_1 = 3.6$ Å , $\Delta\lambda_2 = 6.2$ Å (All 0.6)

Using the Doppler relation and noting that the shift is due to twice the orbital speed: (Factor of two 0.4)

$$v_1 = c\frac{\Delta\lambda_1}{2\lambda_0} = 9.2\times10^{4} \text{ m/s} \quad \text{(0.2)}$$

$$v_2 = c\frac{\Delta\lambda_2}{2\lambda_0} = 1.6\times10^{5} \text{ m/s} \quad \text{(0.2)}$$

The student can use the wavelength of central line and maximum (or minimum) wavelengths. Marking scheme is given in the Excel file.

2.2) As the center of mass is not moving with respect to us: (0.5)

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = 1.7 \quad (0.2)$$

2.3)

Writing $r_i = \frac{v_i}{\omega}$ for $i = 1,2$, we have (0.4)

$r_1 = 3.8 \times 10^9$ m, (0.2) $r_2 = 6.5 \times 10^9$ m (0.2)

2.4)

$r = r_1 + r_2 = 1.0 \times 10^{10}$ m (0.2)

3.1)

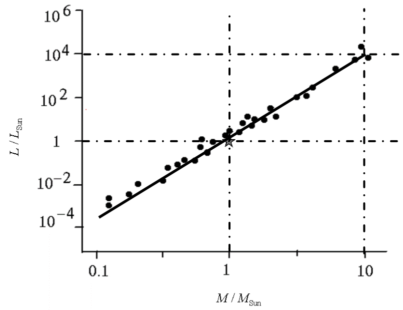
The gravitational force is equal to mass times the centrifugal acceleration

$$G\frac{m_1 m_2}{r^2} = m_1 \frac{v_1^2}{r_1} = m_2 \frac{v_2^2}{r_2} \quad (0.7)$$

Therefore,

$$\begin{cases} m_1 = \dfrac{r^2 v_2^2}{G r_2} \\ m_2 = \dfrac{r^2 v_1^2}{G r_1} \end{cases} \quad (0.1) \quad \Rightarrow \quad \begin{cases} m_1 = 6 \times 10^{30}\,\text{kg} \\ m_2 = 3 \times 10^{30}\,\text{kg} \end{cases} \quad (0.2 + 0.2)$$

4.1) As it is clear from the diagram, with one significant digit, $\alpha = 4$. (0.6)



4.2)

As we have found in the previous section: $L_i = L_{Sun}\left(\frac{M_i}{M_{Sun}}\right)^4$ (0.2)

So,

$$L_1 = 3\times10^{28}\,\text{Watt} \quad (0.2)$$
$$L_2 = 4\times10^{27}\,\text{Watt} \quad (0.2)$$

4.3) The total power of the system is distributed on a sphere with radius d to produce I_0, that is:

$$I_0 = \frac{L_1 + L_2}{4\pi d^2} \quad (0.5) \qquad \Rightarrow d = \sqrt{\frac{L_1 + L_2}{4\pi I_0}} = 1\times10^{18}\,\text{m} \quad (0.2)$$
$$= 100\ \text{ly.} \quad (0.2)$$

4.4) $$\theta \cong \tan\theta = \frac{r}{d} = 1\times10^{-8}\,\text{rad.} \quad (0.2 + 0.2)$$

4.5)

A typical optical wavelength is λ_0. Using uncertainty relation:

$$D = \frac{d\,\lambda_0}{r} \cong 50\ \text{m.} \quad (0.2 + 0.2)$$