Solution

1. The structure of the mortar

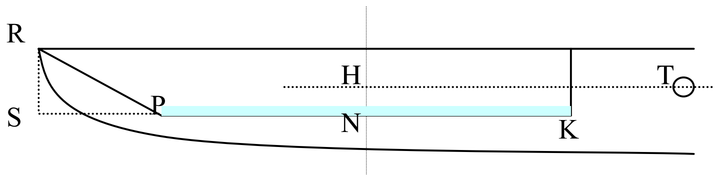
1.1. Calculating the distance TG

The volume of water in the bucket is $V = 1000\,\text{cm}^3 = 10^{-3}\,\text{m}^3$. The length of the bottom of the bucket is $d = L - h\,\mathbf{tan}\,60^0 = \mathbf{(}0.74 - 0.12\,\mathbf{tan}\,60^0\mathbf{)}\,\text{m} = 0.5322\,\text{m}$.

(as the initial data are given with two significant digits, we shall keep only two significant digits in the final answer, but we keep more digits in the intermediate steps).
The height c of the water layer in the bucket is calculated from the formula:

$$V = bcd + b\frac{c}{2}c\tan 60^0 \Rightarrow c = \frac{(d^2 + 2\sqrt{3}V/b)^{1/2} - d}{\sqrt{3}}$$

Inserting numerical values for V, b and d, we find $c = 0.01228\,\text{m}$.
When the lever lies horizontally, the distance, on the horizontal axis, between the rotation axis and the center of mass of water N, is $\text{TH} \approx a + \frac{d}{2} + \frac{c}{4}\mathbf{tan}\,60^\text{o} = 0.4714\,\text{m}$, and $\text{TG} = (m/M)\,\text{TH} = 0.01571\,\text{m}$ (see the figure below).



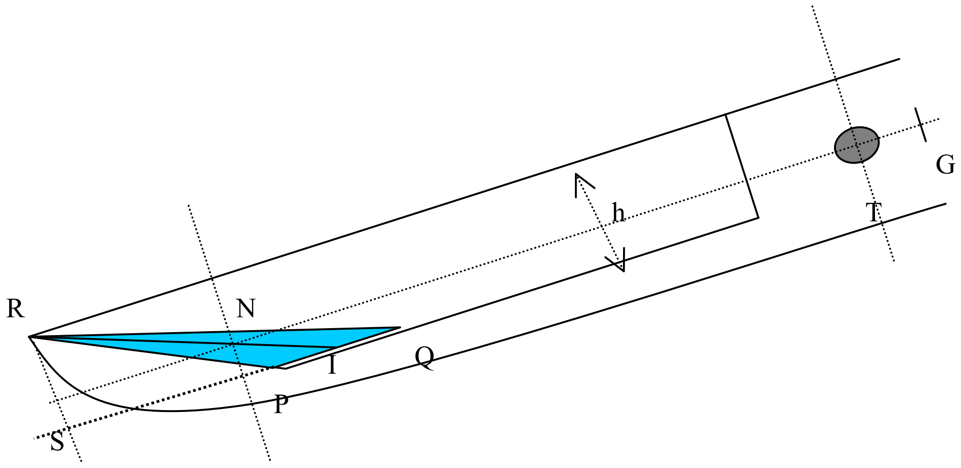
Answer: $\text{TG} = 0.016\,\text{m}$.

1.2. Calculating the values of α_1 and α_2.

When the lever tilts with angle α_1, water level is at the edge of the bucket. At that point the water volume is $10^{-3}\,\text{m}^3$. Assume $\text{PQ} < d$. From geometry $V = hb \times \text{PQ}/2$, from which $\text{PQ} = 0.1111\,\text{m}$. The assumption $\text{PQ} < d$ is obviously satisfied ($d = 0.5322\,\text{m}$).

To compute the angle α_1, we note that $\mathbf{tan}\,\alpha_1 = h/\text{QS}=h/\mathbf{(}\text{PQ}+\sqrt{3}h\mathbf{)}$. From this we find $\alpha_1 = 20.6^\text{o}$.

When the tilt angle is 30°, the bucket is empty**:** $\alpha_2 = 30^\circ$**.**



1.3. Determining the tilt angle β of the lever and the amount of water in the bucket m when the total torque μ on the lever is equal to zero

Denote $\mathrm{PQ} = x(\mathrm{m})$. The amount of water in the bucket is $m = \rho_{\text{water}} \dfrac{xhb}{2} = 9x$ (kg).

$\mu = 0$ when the torque coming from the water in the bucket cancels out the torque coming from the weight of the lever. The cross section of the water in the bucket is the triangle PQR in the figure. The center of mass N of water is located at 2/3 of the meridian RI, therefore NTG lies on a straight line. Then: $mg \times \mathrm{TN} = Mg \times \mathrm{TG}$ or

$$m \times \mathrm{TN} = M \times \mathrm{TG} = 30 \times 0.1571 = 0.4714 \qquad (1)$$

Calculating TN from x then substitute (1) :

$$\mathrm{TN} = L + a - \frac{2}{3}(h\sqrt{3} + \frac{x}{2}) = 0.94 - 0.08\sqrt{3} - \frac{x}{3} = 0.8014 - \frac{x}{3}$$

which implies $m \times \mathrm{TN} = 9x(0.8014 - x/3) = -3x^2 + 7.213x$ (2)

So we find an equation for x:

$$-3x^2 + 7.213x = 0.4714 \qquad (3)$$

The solutions to (3) are $x = 2.337$ and $x = 0.06723$. Since x has to be smaller than 0.5322, we have to take $x = x_0 = 0.06723$ and $m = 9x_0 = 0.6051\,\mathrm{kg}$.

$$\tan\beta = \frac{h}{x + h\sqrt{3}} = 0.4362, \text{ or } \quad \beta = 23.57^\circ.$$

Answer: $m = 0.61\,\mathrm{kg}$ and $\beta = 23.6^\circ$.

2. Parameters of the working mode

2.1.Graphs of $\mu(\alpha)$, $\alpha(t)$, and $\mu(t)$ during one operation cycle**.**

Initially when there is no water in the bucket, $\alpha = 0$, μ has the largest magnitude equal to $gM \times \mathrm{TG} = 30 \times 9.81 \times 0.01571 = 4.624\,\mathrm{N \cdot m}$. Our convention will be that the sign of this torque is negative as it tends to decrease α.

As water flows into the bucket, the torque coming from the water (which carries positive sign) makes μ increase until μ is slightly positive, when the lever starts to lift up. From that moment, by assumption, the amount of water in the bucket is constant.

The lever tilts so the center of mass of water moves away from the rotation axis, leading to an increase of μ, which reaches maximum when water is just about to overflow the edge of the bucket. At this moment $\alpha = \alpha_1 = 20.6^{\mathrm{o}}$.

A simple calculation shows that

$$\mathrm{SI} = \mathrm{SP} + \mathrm{PQ}/2 = 0.12 \times 1.732 + 0.1111/2 = 0.2634\,\mathrm{m}.$$

$$\mathrm{TN} = 0.20 + 0.74 - \frac{2}{3}\mathrm{SI} = 0.7644\,\mathrm{m}.$$

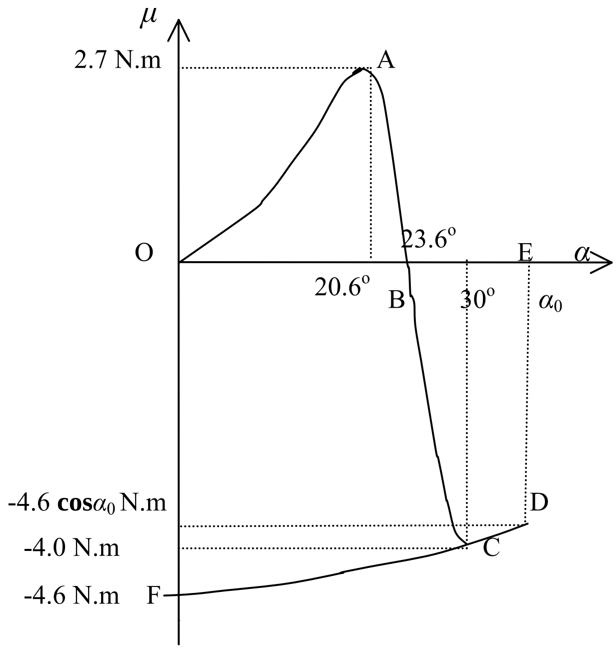
$$\mu_{\max} = (1.0 \times \mathrm{TN} - 30 \times \mathrm{TG})g\cos 20.6^{\mathrm{o}}$$

$$= (1.0 \times 0.7644 - 30 \times 0.01571) \times 9.81 \times \cos 20.6^{\mathrm{o}} = 2.690\ \mathrm{N \cdot m}.$$

Therefore $\mu_{\max} = 2.7\ \mathrm{N \cdot m}$.

As the bucket tilts further, the amount of water in the bucket decreases, and when $\alpha = \beta$, $\mu = 0$. Due to inertia, α keeps increasing and μ keeps decreasing. The bucket is empty when $\alpha = 30^{\mathrm{o}}$, when μ equals $-30 \times g \times \mathrm{TG} \times \cos 30^{\mathrm{o}} = -4.0\,\mathrm{N \cdot m}$. After that α keeps increasing due to inertia to α_0 ($\mu = -gM\,\mathrm{TG}\cos\alpha_0 = -4.62\cos\alpha_0\,\mathrm{N \cdot m}$), then quickly decreases to 0 ($\mu = -4.62\,\mathrm{N \cdot m}$).

On this basis we can sketch the graphs of $\alpha(t)$, $\mu(t)$, and $\mu(\alpha)$ as in the figure below



2.2. The infinitesimal work produced by the torque $\mu(\alpha)$ is $dW = \mu(\alpha)d\alpha$. The energy obtained by the lever during one cycle due to the action of $\mu(\alpha)$ is $W = \oint \mu(\alpha)d\alpha$, which is the area limited by the line $\mu(\alpha)$. Therefore W_{total} is equal to the area enclosed by the curve (OABCDFO) on the graph $\mu(\alpha)$.

The work that the lever transfers to the mortar is the energy the lever receives as it moves from the position $\alpha = \alpha_o$ to the horizontal position $\alpha = 0$. We have W_{pounding} equals to the area of (OEDFO) on the graph $\mu(\alpha)$. It is equal to $gM \times \text{TG} \times \sin\alpha_0 = 4.6\sin\alpha_0$ (J).

2.3. The magnitudes of α_0 can be estimated from the fact that at point D the energy of the lever is zero. We have

area (OABO) = area (BEDCB)

Approximating OABO by a triangle, and BEDCB by a trapezoid, we obtain:

$$23.6 \times 2.7 \times (1/2) = 4.0 \times [(\alpha_0 - 23.6) + (\alpha_0 - 30)] \times (1/2),$$

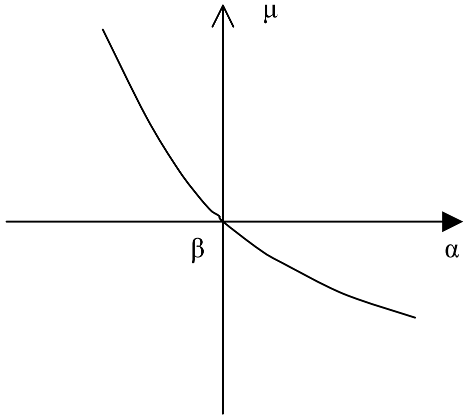
which implies $\alpha_0 = 34.7^{\circ}$. From this we find

$$W_{\text{pounding}} = \text{area (OEDFO)} = \int_{34.76}^{0} -Mg \times \text{TG} \times \cos\alpha\, d\alpha \;= 4.62 \times \sin 34.7^{\circ} = 2.63$$

Thus we find $W_{\text{pounding}} \approx 2.6$ J.

3. The rest mode

3.1.



3.1.1. The bucket is always overflown with water. The two branches of $\mu(\alpha)$ in the vicinity of $\alpha = \beta$ corresponding to increasing and decreasing α coincide with each other.

The graph implies that $\alpha = \beta$ is a stable equilibrium of the mortar.

3.1.2. Find the expression for the torque μ when the tilt angle is $\alpha = \beta + \Delta\alpha$ ($\Delta\alpha$ is small **).**

The mass of water in bucket when the lever tilts with angle α is $m = (1/2)\rho bh\,\mathrm{PQ}$, where $\mathrm{PQ} = h\left(\frac{1}{\tan\alpha} - \frac{1}{\tan 30^0}\right)$**.** A simple calculation shows that when α increases from β to $\beta + \Delta\alpha$, the mass of water increases by $\Delta m = -\frac{bh^2\rho}{2\sin^2\alpha}\Delta\alpha \approx -\frac{bh^2\rho}{2\sin^2\beta}\Delta\alpha$. The torque μ acting on the lever when the tilt is $\beta + \Delta\alpha$ equals the torque due to Δm.

We have $\mu = \Delta m \times g \times \mathrm{TN} \times \cos(\beta + \Delta\alpha)$. TN is found from the equilibrium condition of the lever at tilting angle β:

$$\mathrm{TN} = M \times \mathrm{TG}/m = 30 \times 0.01571/0.605 = 0.779\,\mathrm{m}.$$

We find at the end $\mu = -47.2 \times \Delta\alpha\ \ \mathrm{N\cdot m} \approx -47 \times \Delta\alpha\ \ \mathrm{N\cdot m}$**.**

3.1.3. Equation of motion of the lever

$\mu = I\frac{d^2\alpha}{dt^2}$ where $\mu = -47 \times \Delta\alpha$, $\alpha = \beta + \Delta\alpha$, and I is the sum of moments of inertia of the lever and of the water in bucket relative to the axis T. Here I is not constant the amount of water in the bucket depends on α. When $\Delta\alpha$ is small, one can consider the amount and the shape of water in the bucket to be constant, so I is approximatey a constant. Consider water in bucket as a material point with mass 0.6 kg, a simple calculation gives $I = 12 + 0.6 \times 0.78^2 = 12.36 \approx 12.4\ \mathrm{kg\,m^2}$. We have $-47 \times \Delta\alpha = 12.4 \times \frac{d^2\Delta\alpha}{dt^2}$. That is the equation for a harmonic oscillator with period

$\tau = 2\pi\sqrt{\frac{12.4}{47}} = 3.227$. The answer is therefore $\tau = 3.2\,\mathrm{s}$.

3.2. Harmonic oscillation of lever (around $\alpha = \beta$) when bucket is always overflown. Assume the lever oscillate harmonically with amplitude $\Delta\alpha_0$ around $\alpha = \beta$. At time $t = 0$, $\Delta\alpha = 0$, the bucket is overflown. At time dt the tilt changes by $d\alpha$. We are interested in the case $d\alpha < 0$, i.e., the motion of lever is in the direction of decreasing α, and one needs to add more water to overflow the bucket. The equation of motion is:

$\Delta\alpha = -\Delta\alpha_0 \sin(2\pi t/\tau)$, therefore $d(\Delta\alpha) = d\alpha = -\Delta\alpha_0 (2\pi/\tau)\cos(2\pi t/\tau)dt$.

For the bucket to be overflown, during this time the amount of water falling to the bucket should be at least $dm = -\frac{bh^2\rho}{2\sin^2\beta}d\alpha = \frac{2\Delta\alpha_0 \pi bh^2\rho dt}{2\tau\sin^2\beta}\cos\left(\frac{2\pi t}{\tau}\right)$; dm is maximum at $t = 0$, $dm_0 = \frac{\pi bh^2\rho\Delta\alpha_0}{\tau\sin^2\beta}dt$.

The amount of water falling to the bucket is related to flow rate Φ; $dm_0 = \Phi dt$, therefore $\Phi = \frac{\pi bh^2\rho\Delta\alpha_0}{\tau\sin^2\beta}$.

An overflown bucket is the necessary condition for harmonic oscillations of the lever, therefore the condition for the lever to have harmonic oscillations with ampltude 1^{o} or $2\pi/360\,\mathrm{rad}$ is $\Phi \geq \Phi_1$ with

$$\Phi_1 = \frac{\pi bh^2\rho 2\pi}{360\tau\sin^2\beta} = 0.2309\,\mathrm{kg/s}$$

So $\Phi_1 = 0.23\,\mathrm{kg/s}$.

3.3 Determination of Φ_2

If the bucket remains overflown when the tilt decreases to 20.6^{o}, then the amount of water in bucket should reach 1 kg at this time, and the lever oscillate harmonically with amplitude equal $23.6^{\mathrm{o}} - 20.6^{\mathrm{o}} = 3^{\mathrm{o}}$. The flow should exceed $3\Phi_1$, therefore

$\Phi_2 = 3\times 0.23 \approx 0.7\,\mathrm{kg/s}$.

This is the minimal flow rate for the rice-pounding mortar not to work.

Solution

1.

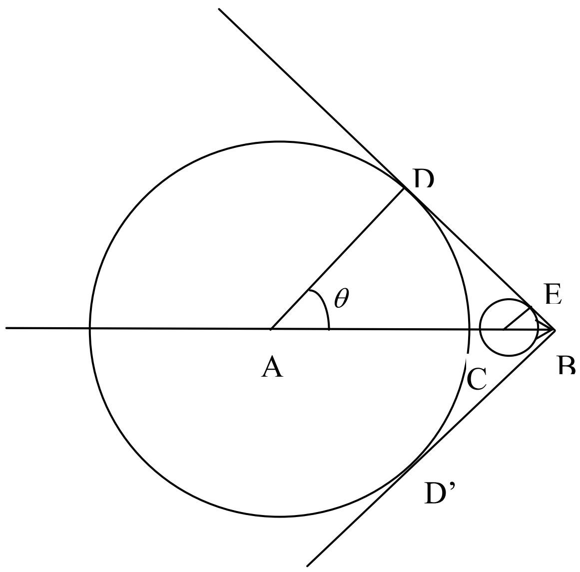


Figure 1

Let us consider a plane containing the particle trajectory. At $t=0$, the particle position is at point A. It reaches point B at $t=t_1$. According to the Huygens principle, at moment $0<t<t_1$, the radiation emitted at A reaches the circle with a radius equal to AD and the one emitted at C reaches the circle of radius CE. The radii of the spheres are proportional to the distance of their centre to B:

$$\frac{\mathrm{CE}}{\mathrm{CB}}=\frac{c\left(t_1-t\right)/n}{v\left(t_1-t\right)}=\frac{1}{\beta n}=\mathrm{const}$$

The spheres are therefore transformed into each other by homothety of vertex B and their envelope is the cone of summit B and half aperture $\varphi=\mathbf{Arcsin}\frac{1}{\beta n}=\frac{\pi}{2}-\theta$, where θ is the angle made by the light ray CE with the particle trajectory.

1.1. The intersection of the wave front with the plane is two straight lines, BD and BD'.

1.2. They make an angle $\varphi=\mathbf{Arcsin}\frac{1}{\beta n}$ with the particle trajectory.

2. The construction for finding the ring image of the particles beam is taken in the plane containing the trajectory of the particle and the optical axis of the mirror.

We adopt the notations:

S – the point where the beam crosses the spherical mirror

F – the focus of the spherical mirror

C – the center of the spherical mirror

IS – the straight-line trajectory of the charged particle making a small angle α with the optical axis of the mirror.

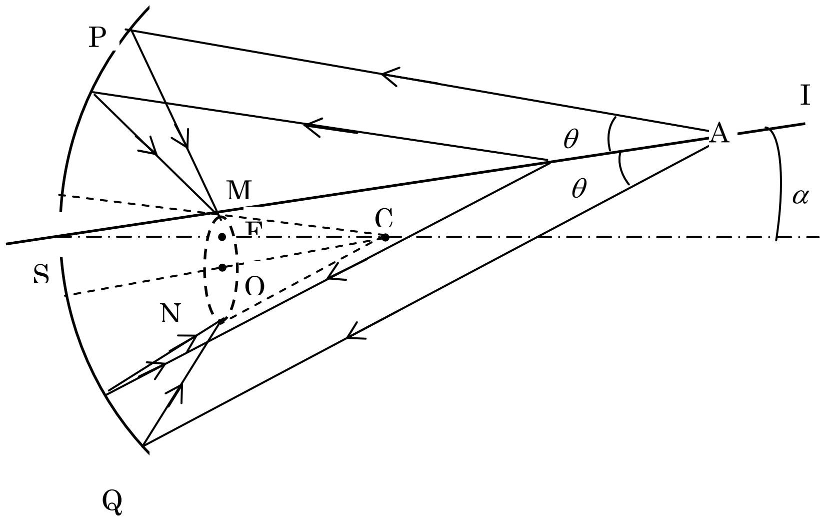


Figure 2

CF = FS = f

CO//IS

CM//AP

CN//AQ

$$\widehat{\mathrm{FCO}} = \alpha \Rightarrow \mathrm{FO} = f \times \alpha$$

$$\widehat{\mathrm{MCO}} = \widehat{\mathrm{OCN}} = \theta \Rightarrow \mathrm{MO} = f \times \theta$$

We draw a straight line parallel to IS passing through the center C. The line intersects the focal plane at O. We have FO $\approx f \times \alpha$.

Starting from C, we draw two lines in both sides of the line CO making with it an angle θ. These two lines intersect the focal plane at M and N, respectively. All the rays of Cherenkov radiation in the plane of the sketch, striking the mirror and being reflected, intersect at M or N.

In three-dimension case, the Cherenkov radiation gives a ring in the focal plane with the center at O (FO $\approx f \times \alpha$) and with the radius MO $\approx f \times \theta$.

In the construction, all the lines are in the plane of the sketch. Exceptionally, the ring is illustrated spatially by a dash line.

3.

3.1. For the Cherenkov effect to occur it is necessary that $n > \frac{c}{v}$, that is $n_{\min} = \frac{c}{v}$.

Putting $\zeta = n - 1 = 2.7 \times 10^{-4} P$, we get

$$\zeta_{\min} = 2.7 \times 10^{-4} P_{\min} = \frac{c}{v} - 1 = \frac{1}{\beta} - 1 \tag{1}$$

Because

$$\frac{Mc^2}{pc} = \frac{Mc}{p} = \frac{Mc}{\frac{Mv}{\sqrt{1-\beta^2}}} = \frac{\sqrt{1-\beta^2}}{\beta} = K \tag{2}$$

then K = 0.094 ; 0.05 ; 0.014 for proton, kaon and pion, respectively.

From (2) we can express β through K as

$$\beta = \frac{1}{\sqrt{1+K^2}} \tag{3}$$

Since $K^2 << 1$ for all three kinds of particles we can neglect the terms of order higher than 2 in K . We get

$$1 - \beta = 1 - \frac{1}{\sqrt{1+K^2}} \approx \frac{1}{2} K^2 = \frac{1}{2}\left(\frac{Mc}{p}\right)^2 \tag{3a}$$

$$\frac{1}{\beta} - 1 = \sqrt{1+K^2} - 1 \approx \frac{1}{2} K^2 = \frac{1}{2}\left(\frac{Mc}{p}\right)^2 \tag{3b}$$

Putting (3b) into (1), we obtain

$$P_{\min} = \frac{1}{2.7\times10^{-4}}\times\frac{1}{2}K^2 \qquad (4)$$

We get the following numerical values of the minimal pressure:

$P_{\min} = 16$ atm for protons,

$P_{\min} = 4.6$ atm for kaons,

$P_{\min} = 0.36$ atm for pions.

3.2. For $\theta_\pi = 2\theta_\kappa$ we have

$$\cos\theta_\pi = \cos 2\theta_\kappa = 2\cos^2\theta_\kappa - 1 \qquad (5)$$

We denote

$$\varepsilon = 1-\beta = 1-\frac{1}{\sqrt{1+K^2}} \approx \frac{1}{2}K^2 \qquad (6)$$

From (5) we obtain

$$\frac{1}{\beta_\pi n} = \frac{2}{\beta_\kappa^2 n^2} - 1 \qquad (7)$$

Substituting $\beta = 1-\varepsilon$ and $n = 1+\zeta$ into (7), we get approximately:

$$\zeta_{\frac{1}{2}} = \frac{4\varepsilon_\kappa - \varepsilon_\pi}{3} = \frac{1}{6}\left(4K_\kappa^2 - K_\pi^2\right) = \frac{1}{6}\left[4.(0.05)^2 - (0.014)^2\right],$$

$$P_{\frac{1}{2}} = \frac{1}{2.7\times10^{-4}}\zeta_{\frac{1}{2}} = 6 \text{ atm}.$$

The corresponding value of refraction index is $n = 1.00162$. We get:

$$\theta_\kappa = 1.6^\circ; \qquad \theta_\pi = 2\theta_\kappa = 3.2^\circ.$$

We do not observe the ring image of protons since

$P_{\frac{1}{2}} = 6\,\text{atm} < 16\,\text{atm} = P_{\min}$ for protons.

4.

4.1. Taking logarithmic differentiation of both sides of the equation $\cos\theta = \frac{1}{\beta n}$, we obtain

$$\frac{\sin\theta \times \Delta\theta}{\cos\theta} = \frac{\Delta\beta}{\beta} \tag{8}$$

Logarithmically differentiating equation (3a) gives

$$\frac{\Delta\beta}{1-\beta} = 2\frac{\Delta p}{p} \tag{9}$$

Combining (8) and (9), taking into account (3b) and putting approximately $\tan\theta = \theta$, we derive

$$\frac{\Delta\theta}{\Delta p} = \frac{2}{\theta} \times \frac{1-\beta}{p\beta} = \frac{K^2}{\theta p} \tag{10}$$

We obtain

-for kaons $K_\kappa = 0.05$, $\theta_\kappa = 1.6^o = 1.6\frac{\pi}{180}\text{rad}$, and so, $\frac{\Delta\theta_\kappa}{\Delta p} = 0.51\frac{1^o}{\text{GeV}/c}$,

-for pions $K_\pi = 0.014$, $\theta_\pi = 3.2^o$, and $\frac{\Delta\theta_\pi}{\Delta p} = 0.02\frac{1^o}{\text{GeV}/c}$.

4.2. $\frac{\Delta\theta_\kappa + \Delta\theta_\pi}{\Delta p} \equiv \frac{\Delta\theta}{\Delta p} = \left(0.51 + 0.02\right)\frac{1^o}{\text{GeV}/c} = 0.53\frac{1^o}{\text{GeV}/c}$.

The condition for two ring images to be distinguishable is $\Delta\theta < 0.1(\theta_\pi - \theta_\kappa) = 0.16^o$.

It follows $\Delta p < \frac{1}{10} \times \frac{1.6}{0.53} = 0.3\ \text{GeV}/c$.

5.

5.1. The lower limit of β giving rise to Cherenkov effect is

$$\beta = \frac{1}{n} = \frac{1}{1.33}. \tag{11}$$

The kinetic energy of a particle having rest mass M and energy E is given by the expression

$$T = E - Mc^2 = \frac{Mc^2}{\sqrt{1-\beta^2}} - Mc^2 = Mc^2\left[\frac{1}{\sqrt{1-\beta^2}} - 1\right]. \tag{12}$$

Substituting the limiting value (11) of β into (12), we get the minimal kinetic energy of the particle for Cherenkov effect to occur:

$$T_{\min} = Mc^2\left[\frac{1}{\sqrt{1-\left(\frac{1}{1.33}\right)^2}} - 1\right] = 0.517\ Mc^2 \tag{13}$$

5.2.

For α particles, $T_{\min} = 0.517 \times 3.8\ \text{GeV} = 1.96\ \text{GeV}$.

For electrons, $T_{\min} = 0.517 \times 0.51\ \text{MeV} = 0.264\ \text{MeV}$.

Since the kinetic energy of the particles emitted by radioactive source does not exceed a few MeV, these are electrons which give rise to Cherenkov radiation in the considered experiment.

6. For a beam of particles having a definite momentum the dependence of the angle θ on the refraction index n of the medium is given by the expression

$$\cos\theta = \frac{1}{n\beta} \tag{14}$$

6.1. Let $\delta\theta$ be the difference of θ between two rings corresponding to two wavelengths limiting the visible range, i.e. to wavelengths of 0.4 μm (violet) and 0.8 μm (red), respectively. The difference in the refraction indexes at these wavelengths is $n_{\mathrm{v}} - n_{\mathrm{r}} = \delta n = 0.02(n-1)$.

Logarithmically differentiating both sides of equation (14) gives

$$\frac{\sin\theta\times\delta\theta}{\cos\theta}=\frac{\delta n}{n} \tag{15}$$

Corresponding to the pressure of the radiator $P = 6$ atm we have from 4.2. the values $\theta_\pi = 3.2^\mathrm{o}$, $n = 1.00162$.

Putting approximately $\tan\theta=\theta$ and $n = 1$, we get $\delta\theta=\frac{\delta n}{\theta}=0.033^\mathrm{o}$.

6.2.

6.2.1. The broadening due to dispersion in terms of half width at half height is, according to (6.1), $\frac{1}{2}\delta\theta=0.017^\mathrm{o}$.

6.2.2. The broadening due to achromaticity is, from 4.1., $0.02\frac{1^\mathrm{o}}{\mathrm{GeV/c}}\times 0.3\ \mathrm{GeV/c}=0.006^\mathrm{o}$, that is three times smaller than above.

6.2.3. The color of the ring changes from red to white then blue from the inner edge to the outer one.

Solution

1. For an altitude change dz , the atmospheric pressure change is :

$$dp = -\rho g dz \qquad (1)$$

where g is the acceleration of gravity, considered constant, ρ is the specific mass of air, which is considered as an ideal gas:

$$\rho = \frac{m}{V} = \frac{p\mu}{RT}$$

Put this expression in (1) :

$$\frac{dp}{p} = -\frac{\mu g}{RT} dz$$

1.1. If the air temperature is uniform and equals T_0, then

$$\frac{dp}{p} = -\frac{\mu g}{RT_0} dz$$

After integration, we have :

$$p(z) = p(0)\mathbf{e}^{-\frac{\mu g}{RT_0}z} \qquad (2)$$

1.2. If

$$T(z) = T(0) - \Lambda z \qquad (3)$$

then

$$\frac{dp}{p} = -\frac{\mu g}{R\left[T(0) - \Lambda z\right]} dz \qquad (4)$$

1.2.1. Knowing that :

$$\int \frac{dz}{T(0) - \Lambda z} = -\frac{1}{\Lambda}\int \frac{d\left[T(0) - \Lambda z\right]}{T(0) - \Lambda z} = -\frac{1}{\Lambda}\mathbf{ln}\left(T(0) - \Lambda z\right)$$

by integrating both members of (4), we obtain :

$$\mathbf{ln}\frac{p(z)}{p(0)} = \frac{\mu g}{R\Lambda}\mathbf{ln}\frac{T(0) - \Lambda z}{T(0)} = \frac{\mu g}{R\Lambda}\mathbf{ln}\left(1 - \frac{\Lambda z}{T(0)}\right)$$

$$p(z) = p(0)\left(1 - \frac{\Lambda z}{T(0)}\right)^{\frac{\mu g}{R\Lambda}} \qquad (5)$$

1.2.2. The free convection occurs if:

$$\frac{\rho(z)}{\rho(0)} > 1$$

The ratio of specific masses can be expressed as follows:

$$\frac{\rho(z)}{\rho(0)} = \frac{p(z)}{p(0)}\frac{T(0)}{T(z)} = \left(1 - \frac{\Lambda z}{T(0)}\right)^{\frac{\mu g}{R\Lambda}-1}$$

The last term is larger than unity if its exponent is negative:

$$\frac{\mu g}{R\Lambda} - 1 < 0$$

Then :

$$\Lambda > \frac{\mu g}{R} = \frac{0.029 \times 9.81}{8.31} = 0.034\ \frac{\mathrm{K}}{\mathrm{m}}$$

2. In vertical motion, the pressure of the parcel always equals that of the surrounding air, the latter depends on the altitude. The parcel temperature T_{parcel} depends on the pressure.

2.1. We can write:

$$\frac{dT_{\text{parcel}}}{dz} = \frac{dT_{\text{parcel}}}{dp}\frac{dp}{dz}$$

p is simultaneously the pressure of air in the parcel and that of the surrounding air.

Expression for $\frac{dT_{\text{parcel}}}{dp}$

By using the equation for adiabatic processes $pV^{\gamma} = \text{const}$ and equation of state, we can deduce the equation giving the change of pressure and temperature in a quasi-equilibrium adiabatic process of an air parcel:

$$T_{\text{parcel}}\, p^{\frac{1-\gamma}{\gamma}} = \text{const} \qquad (6)$$

where $\gamma = \dfrac{c_p}{c_V}$ is the ratio of isobaric and isochoric thermal capacities of air. By logarithmic differentiation of the two members of (6), we have:

$$\frac{dT_{\text{parcel}}}{T_{\text{parcel}}} + \frac{1-\gamma}{\gamma}\frac{dp}{p} = 0$$

Or

$$\frac{dT_{\text{parcel}}}{dp} = \frac{T_{\text{parcel}}}{p}\frac{\gamma-1}{\gamma} \qquad (7)$$

Note: we can use the first law of thermodynamic to calculate the heat received by the parcel in an elementary process: $dQ = \dfrac{m}{\mu}c_V dT_{\text{parcel}} + pdV$, this heat equals zero in an adiabatic process. Furthermore, using the equation of state for air in the parcel $pV = \dfrac{m}{\mu}RT_{\text{parcel}}$ we can derive (6)

Expression for $\dfrac{dp}{dz}$

From (1) we can deduce:

$$\frac{dp}{dz} = -\rho g = -\frac{pg\mu}{RT}$$

where T is the temperature of the surrounding air.

On the basis of these two expressions, we derive the expression for dT_{parcel}/dz :

$$\frac{dT_{\text{parcel}}}{dz} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\frac{T_{\text{parcel}}}{T} = -G \qquad (8)$$

In general, G is not a constant.

2.2.

2.2.1. If at any altitude, $T = T_{\text{parcel}}$, then instead of G in (8), we have :

$$\Gamma = \frac{\gamma-1}{\gamma}\frac{\mu g}{R} = \text{const} \qquad (9)$$

or

$$\Gamma = \frac{\mu g}{c_p} \qquad (9')$$

2.2.2. Numerical value:

$$\Gamma = \frac{1.4-1}{1.4}\frac{0.029\times 9.81}{8.31} = 0.00978\ \frac{\mathrm{K}}{\mathrm{m}} \approx 10^{-2}\ \frac{\mathrm{K}}{\mathrm{m}}$$

2.2.3. Thus, the expression for the temperature at the altitude z in this special atmosphere (called adiabatic atmosphere) is :

$$T(z) = T(0) - \Gamma z \qquad (10)$$

2.3. Search for the expression of $T_{\text{parcel}}(z)$

Substitute T in (7) by its expression given in (3), we have:

$$\frac{dT_{\text{parcel}}}{T_{\text{parcel}}} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\frac{dz}{T(0)-\Lambda z}$$

Integration gives:

$$\ln\frac{T_{\text{parcel}}(z)}{T_{\text{parcel}}(0)} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\left(-\frac{1}{\Lambda}\right)\ln\frac{T(0)-\Lambda z}{T(0)}$$

Finally, we obtain:

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(\frac{T(0)-\Lambda z}{T(0)}\right)^{\frac{\Gamma}{\Lambda}} \qquad (11)$$

2.4.

From (11) we obtain

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(1-\frac{\Lambda z}{T(0)}\right)^{\frac{\Gamma}{\Lambda}}$$

If $\Lambda z << T(0)$, then by putting $x = \frac{-T(0)}{\Lambda z}$, we obtain

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(\left(1+\frac{1}{x}\right)^{x}\right)^{-\frac{\Gamma z}{T(0)}}$$

$$\approx T_{\text{parcel}}(0)\mathbf{e}^{-\frac{\Gamma z}{T(0)}} \approx T_{\text{parcel}}(0)\left(1-\frac{\Gamma z}{T(0)}\right) \approx T_{\text{parcel}}(0) - \Gamma z$$

hence,

$$T_{\text{parcel}}(z) \approx T_{\text{parcel}}(0) - \Gamma z \qquad (12)$$

3. Atmospheric stability

In order to know the stability of atmosphere, we can study the stability of the equilibrium of an air parcel in this atmosphere.

At the altitude z_0, where $T_{\text{parcel}}(z_0) = T(z_0)$, the air parcel is in equilibrium. Indeed, in this case the specific mass ρ of air in the parcel equals ρ'- that of the surrounding air in the atmosphere. Therefore, the buoyant force of the surrounding air on the parcel equals the weight of the parcel. The resultant of these two forces is zero.

Remember that the temperature of the air parcel $T_{\text{parcel}}(z)$ is given by (7), in which we can assume approximately $G = \Gamma$ at any altitude z near $z = z_0$.

Now, consider the stability of the air parcel equilibrium:

Suppose that the air parcel is lifted into a higher position, at the altitude $z_0 + d$ (with d>0), $T_{\text{parcel}}(z_0 + d) = T_{\text{parcel}}(z_0) - \Gamma d$ and $T(z_0 + d) = T(z_0) - \Lambda d$.

- In the case the atmosphere has temperature lapse rate $\Lambda > \Gamma$, we have $T_{\text{parcel}}(z_0 + d) > T(z_0 + d)$, then $\rho < \rho'$. The buoyant force is then larger than the air parcel weight, their resultant is oriented upward and tends to push the parcel away from the equilibrium position.

Conversely, if the air parcel is lowered to the altitude $z_0 - d$ (d>0), $T_{\text{parcel}}(z_0 - d) < T(z_0 - d)$ and then $\rho > \rho'$.

The buoyant force is then smaller than the air parcel weight; their resultant is oriented downward and tends to push the parcel away from the equilibrium position (see Figure 1)

So the equilibrium of the parcel is unstable, and we found that: *An atmosphere with a temperature lapse rate* $\Lambda > \Gamma$ *is unstable.*

- In an atmosphere with temperature lapse rate $\Lambda < \Gamma$, if the air parcel is lifted to a higher position, at altitude $z_0 + d$ (with d>0), $T_{\text{parcel}}(z_0 + d) < T(z_0 + d)$, then

$\rho > \rho'$. The buoyant force is then smaller than the air parcel weight, their resultant is oriented downward and tends to push the parcel back to the equilibrium position.

Conversely, if the air parcel is lowered to altitude $z_0 - d$ (d > 0), $T_{\text{parcel}}(z_0 - d) > T(z_0 - d)$ and then $\rho < \rho'$. The buoyant force is then larger than the air parcel weight, their resultant is oriented upward and tends to push the parcel also back to the equilibrium position (see Figure 2).

So the equilibrium of the parcel is stable, and we found that: *An atmosphere with a temperature lapse rate* $\Lambda < \Gamma$ *is stable.*

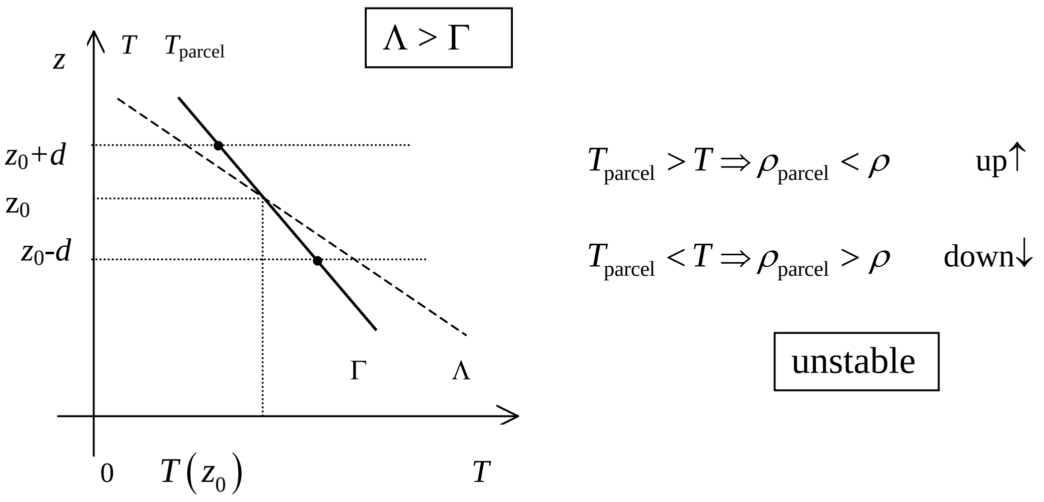


Figure 1

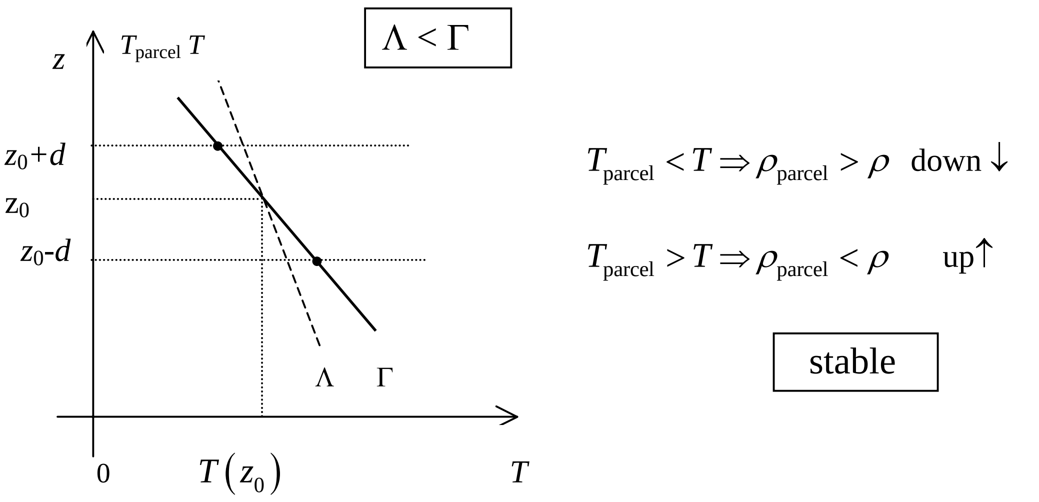


Figure 2

• In an atmosphere with lapse rate $\Lambda = \Gamma$, if the parcel is brought from equilibrium position and put in any other position, it will stay there, the equilibrium is indifferent. *An atmosphere with a temperature lapse rate* $\Lambda = \Gamma$ *is neutral*

3.2. In a stable atmosphere, with $\Lambda < \Gamma$, a parcel, which on ground has temperature $T_{\text{parcel}}(0) > T(0)$ and pressure $p(0)$ equal to that of the atmosphere, can rise and reach a maximal altitude h, where $T_{\text{parcel}}(h) = T(h)$.

In vertical motion from the ground to the altitude h, the air parcel realizes an adiabatic quasi-static process, in which its temperature changes from $T_{\text{parcel}}(0)$ to $T_{\text{parcel}}(h) = T(h)$. Using (11), we can write:

$$\left(1-\frac{\Lambda h}{T(0)}\right)^{-\frac{\Gamma}{\Lambda}} = \frac{T_{\text{parcel}}(0)}{T(h)} = \frac{T_{\text{parcel}}(0)}{T(0)\left(1-\frac{\Lambda h}{T(0)}\right)}$$

$$\left(1-\frac{\Lambda h}{T(0)}\right)^{1-\frac{\Gamma}{\Lambda}} = T_{\text{parcel}}(0)\times T^{-1}(0)$$

$$1-\frac{\Lambda h}{T(0)} = T_{\text{parcel}}^{\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)\times T^{-\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)$$

$$h = \frac{1}{\Lambda}T(0)\left[1-T_{\text{parcel}}^{\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)\times T^{-\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)\right]$$

$$= \frac{1}{\Lambda}\left[T(0)-T_{\text{parcel}}^{-\frac{\Lambda}{\Lambda-\Gamma}}(0)\,T^{\frac{\Gamma}{\Gamma-\Lambda}}(0)\right]$$

So that the maximal altitude h has the following expression:

$$h = \frac{1}{\Lambda}\left[T(0)-\left(\frac{\left(T(0)\right)^{\Gamma}}{\left(T_{\text{parcel}}(0)\right)^{\Lambda}}\right)^{\frac{1}{\Gamma-\Lambda}}\right] \tag{13}$$

4.

Using data from the Table, we obtain the plot of z versus T shown in Figure 3.

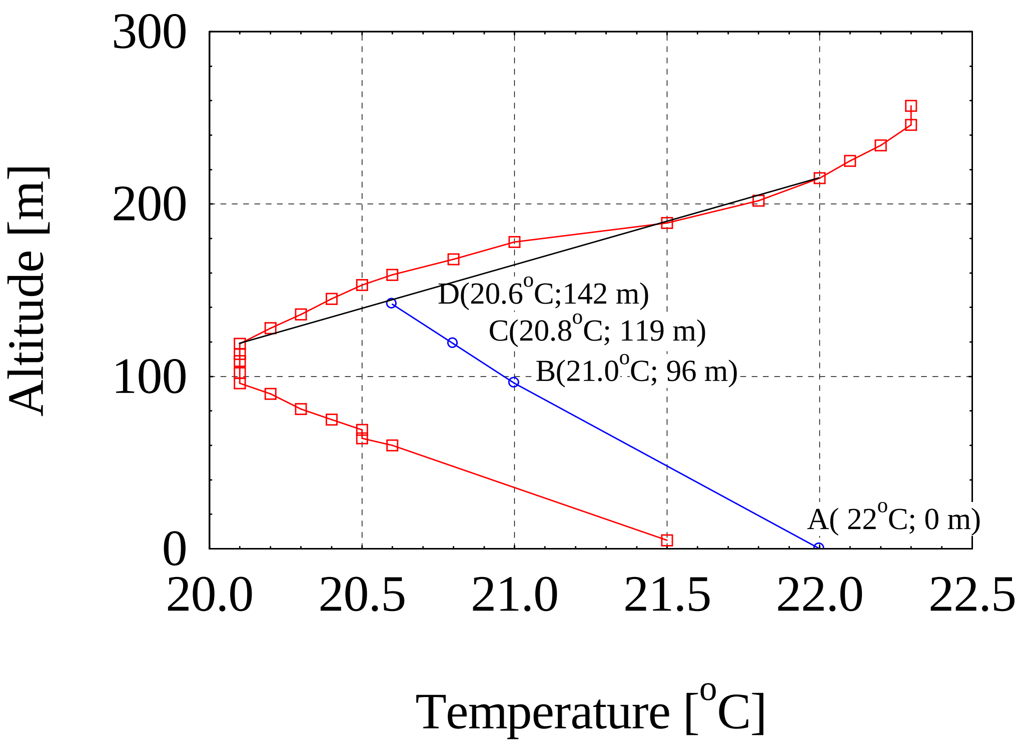


Figure 3

4.1. We can divide the atmosphere under 200m into three layers, corresponding to the following altitudes:

1) $0 < z < 96$ m, $\Lambda_1 = \dfrac{21.5-20.1}{91} = 15.4\times10^{-3}\dfrac{\mathrm{K}}{\mathrm{m}}$.

2) $96\text{ m} < z < 119$ m, $\Lambda_2 = 0$, isothermal layer.

3) $119\text{ m} < z < 215$ m, $\Lambda_3 = -\dfrac{22-20.1}{215-119} = -0.02\dfrac{\mathrm{K}}{\mathrm{m}}$.

In the layer 1), the parcel temperature can be calculated by using (11)

$$T_{\text{parcel}}(96\,\mathrm{m}) = 294.04\text{ K} \approx 294.0\text{ K} \quad \text{that is } 21.0^{\circ}\mathrm{C}$$

In the layer 2), the parcel temperature can be calculated by using its expression in isothermal atmosphere $T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\,\mathbf{exp}\left[-\dfrac{\Gamma z}{T(0)}\right]$.

The altitude 96 m is used as origin, corresponding to 0 m. The altitude 119 m corresponds to 23 m. We obtain the following value for parcel temperature:

$$T_{\text{parcel}}(119\text{ m}) = 293.81\text{ K} \quad \text{that is } 20.8^{\circ}\text{C}$$

4.2. In the layer 3), starting from 119 m, by using (13) we find the maximal elevation h = 23 m, and the corresponding temperature 293.6 K (or 20.6 $^{\circ}$C).

Finally, the mixing height is

$$H = 119 + 23 = 142\text{ m}.$$

And

$$T_{\text{parcel}}(142\text{ m}) = 293.6\text{ K} \quad \text{that is} \quad 20.6^{\circ}\text{C}$$

From this relation, we can find $T_{\text{parcel}}(119\text{ m}) \approx 293.82\text{ K}$ and $h = 23\text{ m}$.

Note: By using approximate expression (12) we can easily find $T_{\text{parcel}}(z) = 294$ K and 293.8 K at elevations 96 m and 119 m, respectively. At 119 m elevation, the difference between parcel and surrounding air temperatures is 0.7 K (= 293.8 – 293.1), so that the maximal distance the parcel will travel in the third layer is $0.7/(\Gamma - \Lambda_3)$ = 0.7/0.03 = 23 m.

5.

Consider a volume of atmosphere of Hanoi metropolitan area being a parallelepiped with height H, base sides L and W. The emission rate of CO gas by motorbikes from 7:00 am to 8:00 am

$$M = 800\,000 \times 5 \times 12 / 3600 = 13\,300\text{ g/s}$$

The CO concentration in air is uniform at all points in the parallelepiped and denoted by $C(t)$.

5.1. After an elementary interval of time dt, due to the emission of the motorbikes, the mass of CO gas in the box increases by Mdt. The wind blows parallel to the short sides W, bringing away an amount of CO gas with mass $LHC(t)udt$. The remaining part raises the CO concentration by a quantity dC in all over the box. Therefore:

$$Mdt - LHC(t)udt = LWHdC$$

or

$$\frac{dC}{dt}+\frac{u}{W}C(t)=\frac{M}{LWH} \qquad (14)$$

5.2. The general solution of (14) is :

$$C(t)=K\,\mathbf{exp}\left(-\frac{ut}{W}\right)+\frac{M}{LHu} \qquad (15)$$

From the initial condition $C(0)=0$, we can deduce :

$$C(t)=\frac{M}{LHu}\left[1-\mathbf{exp}\left(-\frac{ut}{W}\right)\right] \qquad (16)$$

5.3. Taking as origin of time the moment 7:00 am, then 8:00 am corresponds to t =3600 s. Putting the given data in (15), we obtain :

$$C(3600\text{ s})=6.35\times(1-0.64)=2.3\text{ mg/m}^3$$