THEORETICAL PROBLEM No. 1

EVOLUTION OF THE EARTH-MOON SYSTEM

Scientists can determine the distance Earth-Moon with great precision. They achieve this by bouncing a laser beam on special mirrors deposited on the Moon´s surface by astronauts in 1969, and measuring the round travel time of the light (see Figure 1).

Figure 1. A laser beam sent from an observatory is used to measure accurately the distance between the Earth and the Moon.

With these observations, they have directly measured that the Moon is slowly receding from the Earth. That is, the Earth-Moon distance is increasing with time. This is happening because due to tidal torques the Earth is transferring angular momentum to the Moon, see Figure 2. In this problem you will derive the basic parameters of the phenomenon.

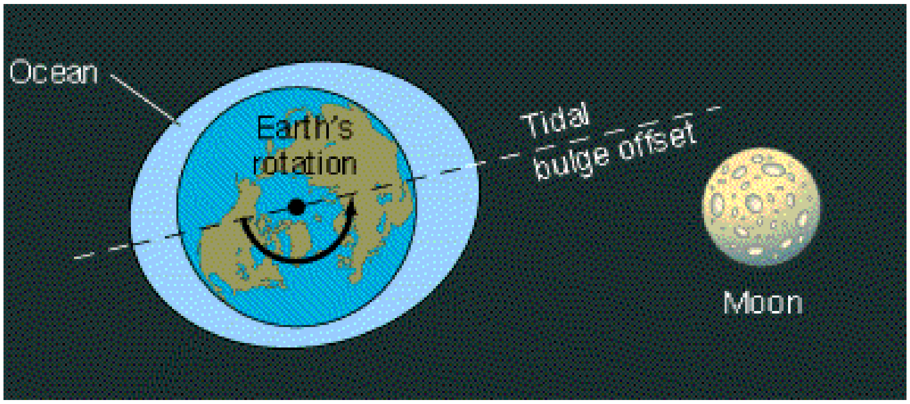


Figure 2. The Moon's gravity produces tidal deformations or "bulges" in the Earth. Because of the Earth's rotation, the line that goes through the bulges is not aligned with the line between the Earth and the Moon. This misalignment produces a torque that transfers angular momentum from the Earth's rotation to the Moon's translation. The drawing is not to scale.

1. Conservation of Angular Momentum.

Let L_1 be the present total angular momentum of the Earth-Moon system. Now, make the following assumptions: i) L_1 is the sum of the rotation of the Earth around its axis and the translation of the Moon in its orbit around the Earth only. ii) The Moon's orbit is circular and the Moon can be taken as a point. iii) The Earth's axis of rotation and the Moon's axis of revolution are parallel. iv) To simplify the calculations, we take the motion to be around the center of the Earth and not the center of mass. Throughout the problem, all moments of inertia, torques and angular momenta are defined around the axis of the Earth. v) Ignore the influence of the Sun.

1a	Write down the equation for the present total angular momentum of the Earth-Moon system. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_{E1}, the present angular frequency of the Earth's rotation; I_{M1}, the present moment of inertia of the Moon with respect to the Earth´s axis; and ω_{M1}, the present angular frequency of the Moon's orbit.	0.2

This process of transfer of angular momentum will end when the period of rotation of the Earth and the period of revolution of the Moon around the Earth have the same duration. At this point the tidal bulges produced by the Moon on the Earth will be aligned with the line between the Moon and the Earth and the torque will disappear.

1b	Write down the equation for the final total angular momentum L_2 of the Earth-Moon system. Make the same assumptions as in Question 1a. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_2, the final angular frequency of the Earth's rotation and Moon's translation; and I_{M2}, the final moment of inertia of the Moon.	0.2

1c	Neglecting the contribution of the Earth´s rotation to the final total angular momentum, write down the equation that expresses the angular momentum conservation for this problem.	0.3

2. Final Separation and Final Angular Frequency of the Earth-Moon System.

Assume that the gravitational equation for a circular orbit (of the Moon around the Earth) is always valid. Neglect the contribution of the Earth´s rotation to the final total angular momentum.

2a	Write down the gravitational equation for the circular orbit of the Moon around the Earth, at the final state, in terms of M_E, ω_2, G and the final separation D_2 between the Earth and the Moon. M_E is the mass of the Earth and G is the gravitational constant.	0.2

2b	Write down the equation for the final separation D_2 between the Earth and the Moon in terms of the known parameters, L_1, the total angular momentum of the system, M_E and M_M, the masses of the Earth and Moon, respectively, and G.	0.5

2c	Write down the equation for the final angular frequency ω_2 of the Earth-Moon system in terms of the known parameters L_1, M_E, M_M and G.	0.5

Below you will be asked to find the numerical values of D_2 and ω_2. For this you need to know the moment of inertia of the Earth.

2d	Write down the equation for the moment of inertia of the Earth I_E assuming it is a sphere with inner density ρ_i from the center to a radius r_i, and with outer density ρ_o from the radius r_i to the surface at a radius r_o (see Figure 3).	0.5

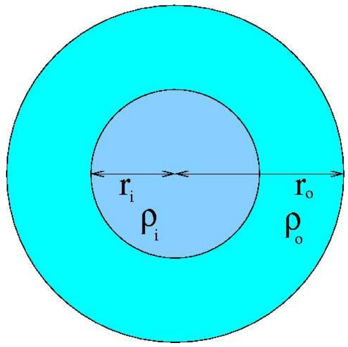


Figure 3. The Earth as a sphere with two densities, ρ_i and ρ_o.

Determine the numerical values requested in this problem always to *two significant digits*.

2e	Evaluate the moment of inertia of the Earth I_E, using $\rho_i = 1.3\times10^4$ kg m^{-3}, $r_i = 3.5\times10^6$ m, $\rho_o = 4.0\times10^3$ kg m^{-3}, and $r_o = 6.4\times10^6$ m.	0.2

The masses of the Earth and Moon are $M_E = 6.0\times10^{24}$ kg and $M_M = 7.3\times10^{22}$ kg, respectively. The present separation between the Earth and the Moon is $D_1 = 3.8\times10^8$ m. The present angular frequency of the Earth's rotation is $\omega_{E1} = 7.3\times10^{-5}$ s^{-1}. The present angular frequency of the Moon's translation around the Earth is $\omega_{M1} = 2.7\times10^{-6}$ s^{-1}, and the gravitational constant is $G = 6.7\times10^{-11}$ m^3 kg^{-1} s^{-2}.

2f	Evaluate the numerical value of the total angular momentum of the system, L_1.	0.2

2g	Find the final separation D_2 in meters and in units of the present separation D_1.	0.3

2h	Find the final angular frequency ω_2 in s^{-1}, as well as the final duration of the day in units of present days.	0.3

Verify that the assumption of neglecting the contribution of the Earth´s rotation to the final total angular momentum is justified by finding the ratio of the final angular momentum of the Earth to that of the Moon. This should be a small quantity.

2i	Find the ratio of the final angular momentum of the Earth to that of the Moon.	0.2

3. How much is the Moon receding per year?

Now, you will find how much the Moon is receding from the Earth each year. For this, you will need to know the equation for the torque acting at present on the Moon. Assume that the tidal bulges can be approximated by two point masses, each of mass m, located on the surface of the Earth, see Fig. 4. Let θ be the angle between the line that goes through the bulges and the line that joins the centers of the Earth and the Moon.

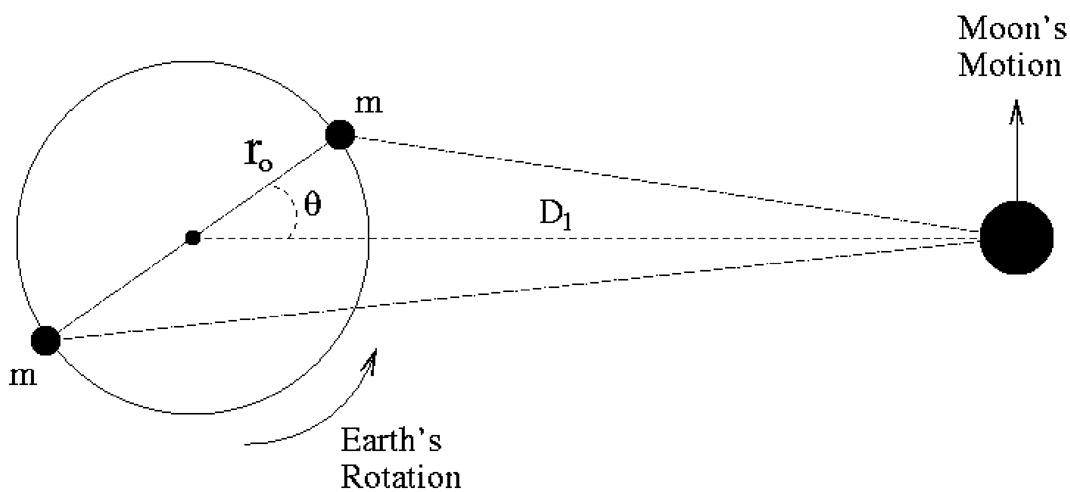


Figure 4. Schematic diagram to estimate the torque produced on the Moon by the bulges on the Earth. The drawing is not to scale.

3a	Find F_c, the magnitude of the force produced on the Moon by the closest point mass.	0.4

3b	Find F_f, the magnitude of the force produced on the Moon by the farthest point mass.	0.4

You may now evaluate the torques produced by the point masses.

3c	Find the magnitude of τ_c, the torque produced by the closest point mass.	0.4

3d	Find the magnitude of τ_f, the torque produced by the farthest point mass.	0.4

3e	Find the magnitude of the total torque τ produced by the two masses. Since $r_o << D_1$ you should approximate your expression to lowest significant order in r_o / D_1. You may use that $(1+x)^a \approx 1+ax$, if $x<<1$.	1.0

3f	Calculate the numerical value of the total torque τ, taking into account that $\theta = 3^\circ$ and that $m = 3.6 \times 10^{16}$ kg (note that this mass is of the order of 10^{-8} times the mass of the Earth).	0.5

Since the torque is the rate of change of angular momentum with time, find the increase in the distance Earth-Moon at present, per year. For this step, express the angular momentum of the Moon in terms of M_M, M_E, D_1 and G only.

3g	Find the increase in the distance Earth-Moon at present, per year.	1.0

Finally, estimate how much the length of the day is increasing each year.

3h	Find the decrease of ω_{E1} per year and how much is the length of the day at present increasing each year.	1.0

4. Where is the energy going?

In contrast to the angular momentum, that is conserved, the total (rotational plus gravitational) energy of the system is not. We will look into this in this last section.

4a	Write down an equation for the total (rotational plus gravitational) energy of the Earth-Moon system at present, E. Put this equation in terms of I_E, ω_{E1}, M_M, M_E, D_1 and G only.	0.4

4b	Write down an equation for the change in E, ΔE, as a function of the changes in D_1 and in ω_{E1}. Evaluate the numerical value of ΔE for a year, using the values of changes in D_1 and in ω_{E1} found in questions 3g and 3h.	0.4

Verify that this loss of energy is consistent with an estimate for the energy dissipated as heat in the tides produced by the Moon on the Earth. Assume that the tides rise, on the average by 0.5 m, a layer of water $h =$ 0.5 m deep that covers the surface of the Earth (for simplicity assume that all the surface of the Earth is covered with water). This happens twice a day. Further assume that 10% of this gravitational energy is dissipated as heat due to viscosity when the water descends. Take the density of water to be $\rho_{water} = 10^3$ kg m^{-3}, and the gravitational acceleration on the surface of the Earth to be $g = 9.8$ m s^{-2}.

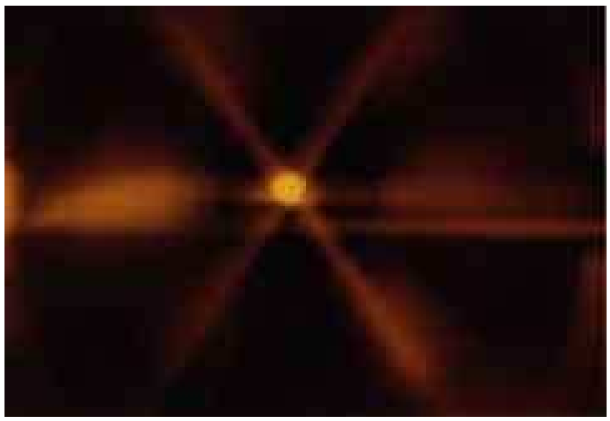
4c	What is the mass of this surface layer of water?	0.2

4d	Calculate how much energy is dissipated in a year? How does this compare with the energy lost per year by the Earth-Moon system at present?	0.3

THEORETICAL PROBLEM 2

DOPPLER LASER COOLING AND OPTICAL MOLASSES

The purpose of this problem is to develop a simple theory to understand the so-called "laser cooling" and "optical molasses" phenomena. This refers to the cooling of a beam of neutral atoms, typically alkaline, by counterpropagating laser beams with the same frequency. This is part of the Physics Nobel Prize awarded to S. Chu, P. Phillips and C. Cohen-Tannoudji in 1997.



The image above shows sodium atoms (the bright spot in the center) trapped at the intersection of three orthogonal pairs of opposing laser beams. The trapping region is called "optical molasses" because the dissipative optical force resembles the viscous drag on a body moving through molasses.

In this problem you will analyze the basic phenomenon of the interaction between a photon incident on an atom and the basis of the dissipative mechanism in one dimension.

PART I: BASICS OF LASER COOLING

Consider an atom of mass m moving in the $+x$ direction with velocity v. For simplicity, we shall consider the problem to be one-dimensional, namely, we shall ignore the y and z directions (see figure 1). The atom has two internal energy levels. The energy of the lowest state is considered to be zero and the energy of the excited state to be $\hbar\omega_0$, where $\hbar = h/2\pi$. The atom is initially in the lowest state. A laser beam with frequency ω_L in the laboratory is directed in the $-x$ direction and it is incident on the atom. Quantum mechanically the laser is composed of a large number of photons, each with energy $\hbar\omega_L$ and momentum $-\hbar q$. A photon can be absorbed by the atom and later spontaneously emitted; this emission can occur with equal probabilities along the $+x$ and $-x$ directions. Since the atom moves at non-relativistic speeds, $v/c << 1$ (with c the speed of light) keep terms up to first order in this quantity only. Consider also $\hbar q / mv << 1$, namely, that the momentum of the atom is much larger than the momentum of a single photon. In writing your answers, keep only corrections linear in either of the above quantities.

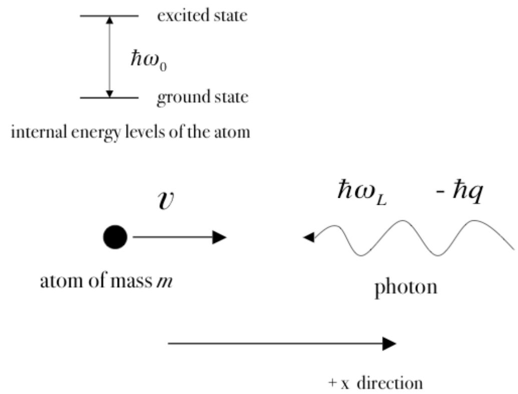


Fig.1 Sketch of an atom of mass m with velocity v in the $+x$ direction, colliding with a photon with energy $\hbar\omega_L$ and momentum $-\hbar q$. The atom has two internal states with energy difference $\hbar\omega_0$.

Assume that the laser frequency ω_L is tuned such that, as seen by the moving atom, it is in resonance with the internal transition of the atom. Answer the following questions:

1. Absorption.

1a	Write down the resonance condition for the absorption of the photon.	0.2

1b	Write down the momentum p_{at} of the atom after absorption, as seen in the laboratory.	0.2

1c	Write down the total energy ε_{at} of the atom after absorption, as seen in the laboratory.	0.2

2. Spontaneous emission of a photon in the $-x$ direction.

At some time after the absorption of the incident photon, the atom may emit a photon in the $-x$ direction.

2a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2c	Write down the momentum of the atom p_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2d	Write down the total energy of the atom ε_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

3. Spontaneous emission of a photon in the $+x$ direction.

At some time after the absorption of the incident photon, the atom may instead emit a photon in the $+x$ direction.

3a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3c	Write down the momentum of the atom p_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3d	Write down the total energy of the atom ε_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

4. Average emission after the absorption.

The spontaneous emission of a photon in the $-x$ or in the $+x$ directions occurs with the same probability. Taking this into account, answer the following questions.

4a	Write down the average energy of an emitted photon, ε_{ph}, after the emission process.	0.2

4b	Write down the average momentum of an emitted photon p_{ph}, after the emission process.	0.2

4c	Write down the average total energy of the atom ε_{at}, after the emission process.	0.2

4d	Write down the average momentum of the atom p_{at}, after the emission process.	0.2

5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser radiation and the atom.

5a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.2

5b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.2

6. Energy and momentum transfer by a laser beam along the $+x$ direction.

Consider now that a laser beam of frequency ω_L' is incident on the atom along the $+x$ direction, while the atom moves also in the $+x$ direction with velocity v. Assuming a resonance condition between the internal transition of the atom and the laser beam, as seen by the atom, answer the following questions:

6a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.3

6b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.3

PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Nature, however, imposes an inherent uncertainty in quantum processes. Thus, the fact that the atom can spontaneously emit a photon in a *finite* time after absorption, gives as a result that the resonance condition does not have to be obeyed *exactly* as in the discussion above. That is, the frequency of the laser beams ω_L and ω_L' may have any value and the absorption-emission process can still occur. These will happen with different (quantum) probabilities and, as one should expect, the maximum probability is found at the exact resonance condition. On the average, the time elapsed between a single process of absorption and emission is called the lifetime of the excited energy level of the atom and it is denoted by Γ^{-1}.

Consider a collection of N atoms at *rest* in the laboratory frame of reference, and a laser beam of frequency ω_L incident on them. The atoms absorb and emit continuously such that there is, on average, N_{exc} atoms in the excited state (and therefore, $N - N_{exc}$ atoms in the ground state). A quantum mechanical calculation yields the following result:

$$N_{exc} = N \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where ω_0 is the resonance frequency of the atomic transition and Ω_R is the so-called Rabi frequency; Ω_R^2 is proportional to the *intensity* of the laser beam. As mentioned above, you can see that this number is different from zero even if the resonance frequency ω_0 is different from the frequency of the laser beam ω_L. An alternative way of expressing the previous result is that the number of absorption-emission processes per unit of time is $N_{exc}\Gamma$.

Consider the physical situation depicted in Figure 2, in which two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

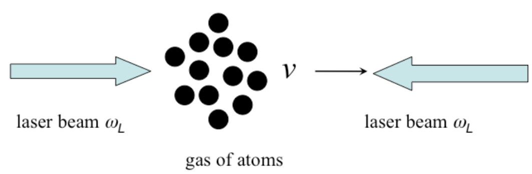


Figure 2. Two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

7. Force on the atomic beam by the lasers.

7a	With the information found so far, find the force that the lasers exert on the atomic beam. You should assume that $mv >> \hbar q$.	1.5

8. Low velocity limit.

Assume now that the velocity of the atoms is small enough, such that you can expand the force up to first order in v.

8a	Find an expression for the force found in Question (7a), in this limit.	1.5

Using this result, you can find the conditions for speeding up, slowing down, or no effect at all on the atoms by the laser radiation.

8b	Write down the condition to obtain a positive force (speeding up the atoms).	0.25

8c	Write down the condition to obtain a zero force.	0.25

8d	Write down the condition to obtain a negative force (slowing down the atoms).	0.25

8e	Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms.	0.25

9. Optical molasses.

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, at $t=0$, the gas of atoms has velocity v_0.

9a	In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time τ.	1.5

9b	Assume now that the gas of atoms is in thermal equilibrium at a temperature T_0. Find the temperature T after the laser beams have been on for a time τ.	0.5

This model does not allow you to go to arbitrarily low temperatures.

THEORETICAL PROBLEM No. 3

WHY ARE STARS SO LARGE?

The stars are spheres of hot gas. Most of them shine because they are fusing hydrogen into helium in their central parts. In this problem we use concepts of both classical and quantum mechanics, as well as of electrostatics and thermodynamics, to understand why stars have to be big enough to achieve this fusion process and also derive what would be the mass and radius of the smallest star that can fuse hydrogen.

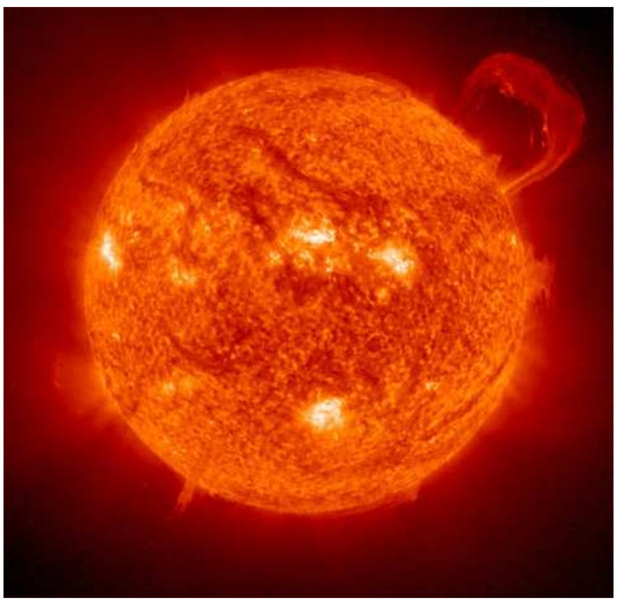


Figure 1. Our Sun, as most stars, shines as a result of thermonuclear fusion of hydrogen into helium in its central parts.

USEFUL CONSTANTS

Gravitational constant = $G = 6.7\times10^{-11}$ m^3 kg^{-1} s^2

Boltzmann´s constant = $k = 1.4\times10^{-23}$ J K^{-1}

Planck's constant = $h = 6.6\times10^{-34}$ m^2 kg s^{-1}

Mass of the proton = $m_p = 1.7\times10^{-27}$ kg

Mass of the electron = $m_e = 9.1\times10^{-31}$ kg

Unit of electric charge = $q = 1.6\times10^{-19}$ C

Electric constant (vacuum permittivity) = $\varepsilon_0 = 8.9\times10^{-12}$ C^2 N^{-1} m^{-2}

Radius of the Sun = $R_S = 7.0\times10^8$ m

Mass of the Sun = $M_S = 2.0\times10^{30}$ kg

1. **A classical estimate of the temperature at the center of the stars.**

Assume that the gas that forms the star is pure ionized hydrogen (electrons and protons in equal amounts), and that it behaves like an ideal gas. From the classical point of view, to fuse two protons, they need to get as close as 10^{-15} m for the short range strong nuclear force, which is attractive, to become dominant. However, to bring them together they have to overcome first the repulsive action of Coulomb's force. Assume classically that the two protons (taken to be point sources) are moving in an antiparallel way, each with velocity v_{rms}, the root-mean-square (rms) velocity of the protons, in a one-dimensional frontal collision.

1a	What has to be the temperature of the gas, T_c, so that the distance of closest approach of the protons, d_c, equals 10^{-15} m? Give this and all numerical values in this problem up to two significant figures.	1.5

2. **Finding that the previous temperature estimate is wrong.**

To check if the previous temperature estimate is reasonable, one needs an independent way of estimating the central temperature of a star. The structure of the stars is very complicated, but we can gain significant understanding making some assumptions. Stars are in equilibrium, that is, they do not expand or contract because the inward force of gravity is balanced by the outward force of pressure (see Figure 2). For a slab of gas the equation of hydrostatic equilibrium at a given distance r from the center of the star, is given by

$$\frac{\Delta P}{\Delta r} = -\frac{G M_r \rho_r}{r^2},$$

where P is the pressure of the gas, G the gravitational constant, M_r the mass of the star within a sphere of radius r, and ρ_r is the density of the gas in the slab.

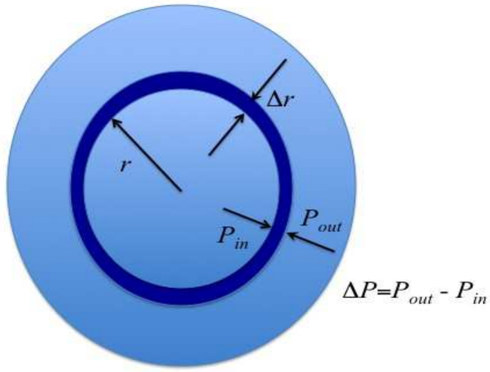


Figure 2. The stars are in hydrostatic equilibrium, with the pressure difference balancing gravity.

An order of magnitude estimate of the central temperature of the star can be obtained with values of the parameters at the center and at the surface of the star, making the following approximations:

$\Delta P \approx P_o - P_c$,

where P_c and P_o are the pressures at the center and surface of the star, respectively. Since $P_c >> P_o$, we can assume that

$\Delta P \approx -P_c$.

Within the same approximation, we can write

$\Delta r \approx R$,

where R is the total radius of the star, and

$M_r \approx M_R = M$,

with M the total mass of the star.

The density may be approximated by its value at the center,

$\rho_r \approx \rho_c$.

You can assume that the pressure is that of an ideal gas.

2a	Find an equation for the temperature at the center of the star, T_c, in terms of the radius and mass of the star and of physical constants only.	0.5

We can use now the following prediction of this model as a criterion for its validity:

2b	Using the equation found in (2a) write down the ratio M/R expected for a star in terms of physical constants and T_c only.	0.5

2c	Use the value of T_c derived in section (1a) and find the numerical value of the ratio M/R expected for a star.	0.5

2d	Now, calculate the ratio $M(Sun)/R(Sun)$, and verify that this value is much smaller than the one found in (2c).	0.5

3. A quantum mechanical estimate of the temperature at the center of the stars

The large discrepancy found in (2d) suggests that the classical estimate for T_c obtained in (1a) is not correct. The solution to this discrepancy is found when we consider quantum mechanical effects, that tell us that the protons behave as waves and that a single proton is smeared on a size of the order of λ_p, the de Broglie wavelength. This implies that if d_c, the distance of closest approach of the protons is of the order of λ_p, the protons in a quantum mechanical sense overlap and can fuse.

3a	Assuming that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition that allows fusion, for a proton with velocity v_{rms}, find an equation for T_c in terms of physical constants only.	1.0

3b	Evaluate numerically the value of T_c obtained in (3a).	0.5

3c	Use the value of T_c derived in (3b) to find the numerical value of the ratio M/R expected for a star, using the formula derived in (2b). Verify that this value is quite similar to the ratio $M(Sun)/R(Sun)$ observed.	0.5

Indeed, stars in the so-called *main sequence* (fusing hydrogen) approximately do follow this ratio for a large range of masses.

4. **The mass/radius ratio of the stars.**

The previous agreement suggests that the quantum mechanical approach for estimating the temperature at the center of the Sun is correct.

4a	Use the previous results to demonstrate that for any star fusing hydrogen, the ratio of mass M to radius R is the same and depends only on physical constants. Find the equation for the ratio M/R for stars fusing hydrogen.	0.5

5. **The mass and radius of the smallest star.**

The result found in (4a) suggests that there could be stars of any mass as long as such a relationship is fulfilled; however, this is not true.

The gas inside normal stars fusing hydrogen is known to behave approximately as an ideal gas. This means that d_e, the typical separation *between electrons* is on the average larger that λ_e, their typical de Broglie wavelength. If closer, the electrons would be in a so-called degenerate state and the stars would behave differently. Note the distinction in the ways we treat protons and electrons inside the star. For protons, their de Broglie waves should overlap closely as they collide in order to fuse, whereas for electrons their de Broglie waves should not overlap in order to remain as an ideal gas.

The density in the stars increases with decreasing radius. Nevertheless, for this order-of-magnitude estimate assume they are of uniform density. You may further use that $m_p >> m_e$.

5a	Find an equation for n_e, the average electron number density inside the star.	0.5

5b	Find an equation for d_e, the typical separation between electrons inside the star.	0.5

5c	Use the $d_e \geq \frac{\lambda_e}{2^{1/2}}$ condition to write down an equation for the radius of the smallest normal star possible. Take the temperature at the center of the star as typical for all the stellar interior.	1.5

5d	Find the numerical value of the radius of the smallest normal star possible, both in meters and in units of solar radius.	0.5

5e	Find the numerical value of the mass of the smallest normal star possible, both in kg and in units of solar masses.	0.5

6. Fusing helium nuclei in older stars.

As stars get older they will have fused most of the hydrogen in their cores into helium (He), so they are forced to start fusing helium into heavier elements in order to continue shining. A helium nucleus has two protons and two neutrons, so it has twice the charge and approximately four times the mass of a proton. We saw before that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition for the protons to fuse.

6a	Set the equivalent condition for helium nuclei and find $v_{rms}(He)$, the rms velocity of the helium nuclei and $T(He)$, the temperature needed for helium fusion.	0.5