THEORETICAL PROBLEM No. 1

EVOLUTION OF THE EARTH-MOON SYSTEM

SOLUTIONS

1. Conservation of Angular Momentum

1a	$L_1 = I_E\omega_{E1} + I_{M1}\omega_{M1}$	0.2

1b	$L_2 = I_E\omega_2 + I_{M2}\omega_2$	0.2

1c	$I_E\omega_{E1} + I_{M1}\omega_{M1} = I_{M2}\omega_2 = L_1$	0.3

2. Final Separation and Angular Frequency of the Earth-Moon System.

2a	$\omega_2^2 D_2^3 = GM_E$	0.2

2b	$D_2 = \dfrac{L_1^2}{GM_E M_M^2}$	0.5

2c	$\omega_2 = \dfrac{G^2 M_E^2 M_M^3}{L_1^3}$	0.5

2d	The moment of inertia of the Earth will be the addition of the moment of inertia of a sphere with radius r_o and density ρ_o and of a sphere with radius r_i and density $\rho_i - \rho_o$: $I_E = \dfrac{2}{5}\dfrac{4\pi}{3}[r_o^5\rho_o + r_i^5(\rho_i - \rho_o)]$.	0.5

2e	$I_E = \dfrac{2}{5}\dfrac{4\pi}{3}[r_o^5\rho_o + r_i^5(\rho_i - \rho_o)] = 8.0\times10^{37}$ kg m^2	0.2

2f	$L_1 = I_E\omega_{E1} + I_{M1}\omega_{M1} = 3.4\times10^{34}$ kg m^2 s^{-1}	0.2

2g	$D_2 = 5.4\times10^{8}$ m, that is $D_2 = 1.4 D_1$	0.3

2h	$\omega_2 = 1.6\times10^{-6}$ s^{-1}, that is, a period of 46 days.	0.3

2i	Since $I_E\,\omega_2 = 1.3\times10^{32}$ kg m^2 s^{-1} and $I_{M2}\,\omega_2 = 3.4\times10^{34}$ kg m^2 s^{-1}, the approximation is justified since the final angular momentum of the Earth is 1/260 of that of the Moon.	0.2

3. How much is the Moon receding per year?

3a	Using the law of cosines, the magnitude of the force produced by the mass *m* closest to the Moon will be: $F_c = \frac{G\,m\,M_M}{D_1^2 + r_o^2 - 2 D_1 r_o \cos(\theta)}$	0.4

3b	Using the law of cosines, the magnitude of the force produced by the mass *m* farthest to the Moon will be: $F_f = \frac{G\,m\,M_M}{D_1^2 + r_o^2 + 2 D_1 r_o \cos(\theta)}$	0.4

3c	Using the law of sines, the torque will be $\tau_c = F_c \frac{\sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 - 2\, D_1\, r_o \cos(\theta)]^{1/2}} = \frac{G\,m\,M_M \sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 - 2 D_1 r_o \cos(\theta)]^{3/2}}$	0.4

3d	Using the law of sines, the torque will be $\tau_f = F_f \frac{\sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 + 2\, D_1\, r_o \cos(\theta)]^{1/2}} = \frac{G\,m\,M_M \sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 + 2 D_1 r_o \cos(\theta)]^{3/2}}$	0.4

3e	$\tau_c - \tau_f = G\,m\,M_M \sin(\theta)\, r_0\, D_1^{-2}\,(1 - \frac{3r_o^2}{2D_1^2} + \frac{3r_o\cos(\theta)}{D_1} - 1 + \frac{3r_o^2}{2D_1^2} + \frac{3r_o\cos(\theta)}{D_1})$ $= \frac{6\,G\,m\,M_M\, r_o^2 \sin(\theta)\cos(\theta)}{D_1^3}$	1.0

3f	$\tau = \dfrac{6 G m M_M r_o^2 \sin(\theta)\cos(\theta)}{D_1^3} = 4.1\times 10^{16}$ N m	0.5

3g	Since $\omega_{M1}^2 D_1^3 = G M_E$, we have that the angular momentum of the Moon is $I_{M1}\,\omega_{M1} = M_M\, D_1^2 \left[\dfrac{G\, M_E}{D_1^3}\right]^{1/2} = M_M\,\left[D_1\, G\, M_E\right]^{1/2}$ The torque will be: $\tau = \dfrac{M_M \left[G M_E\right]^{1/2}\, \Delta(D_1^{1/2})}{\Delta t} = \dfrac{M_M\, \left[G M_E\right]^{1/2}\ \ \Delta D_1}{2\left[D_1\right]^{1/2} \Delta t}$ So, we have that $\Delta D_1 = \dfrac{2\,\tau\,\Delta t}{M_M}\left[\dfrac{D_1}{G M_E}\right]^{1/2}$ That for $\Delta t = 3.1\times 10^7$ s = 1 year, gives $\Delta D_1 = 0.034$ m. This is the yearly increase in the Earth-Moon distance.	1.0

3h	We now use that $\tau = -\dfrac{I_E\, \Delta\omega_{E1}}{\Delta t}$ from where we get $\Delta\omega_{E1} = -\dfrac{\tau\,\Delta t}{I_E}$ that for $\Delta t = 3.1\times 10^7$ s = 1 year gives $\Delta\omega_{E1} = -1.6\times 10^{-14}\,\text{s}^{-1}$. If P_E is the period of time considered, we have that: $\dfrac{\Delta P_E}{P_E} = -\dfrac{\Delta\omega_{E1}}{\omega_E}$ since $P_E = 1\, day = 8.64\times 10^4$ s, we get $\Delta P_E = 1.9\times 10^{-5}$ s. This is the amount of time that the day lengthens in a year.	1.0

4. Where is the energy going?

4a	The present total (rotational plus gravitational) energy of the system is: $E = \dfrac{1}{2} I_E\,\omega_{E1}^2 + \dfrac{1}{2} I_M\,\omega_{M1}^2 - \dfrac{G M_E M_M}{D_1}$. Using that $\omega_{M1}^2\, D_1^3 = G\, M_E$, we get	0.4

	$E=\frac{1}{2}I_E\,\omega_{E1}^2-\frac{1}{2}\frac{GM_E\,M_M}{D_1}$	

4b	$\Delta E=I_E\,\omega_{E1}\,\Delta\omega_{E1}+\frac{1}{2}\frac{GM_E\,M_M}{D_1^2}\Delta D_1$, that gives $\Delta E=-9.0\times10^{19}$ J	0.4

4c	$M_{water}=4\pi r_o^2\times h\times\rho_{water}$ kg = 2.6×10^{17} kg.	0.2

4d	$\Delta E_{water}=-g\,M_{water}\times0.5\,m\times2\,day^{-1}\times365\,days\times0.1=-9.3\times10^{19}$ J. Then, the two energy estimates are comparable.	0.3

THEORETICAL PROBLEM 2

SOLUTION

DOPPLER LASER COOLING AND OPTICAL MOLASSES

The key to this problem is the Doppler effect (to be precise, the longitudinal Doppler effect): The frequency of a monochromatic beam of light detected by an observer depends on its state of motion relative to the emitter, i.e. the observed frequency is

$$\omega' = \omega\sqrt{\frac{1 \pm v/c}{1 \mp v/c}} \approx \omega\left(1 \pm \frac{v}{c}\right)$$

where v is the relative speed of emitter and observer and ω the frequency of the emitter. The upper-lower signs correspond, respectively, when source and observer move towards or away from each other. The second equality holds in the limit of low velocities (non-relativistic limit).

The frequency of the laser in the lab is ω_L; ω_0 is the transition frequency of the atom; the atom moves with speed v towards the incident direction of the laser:

It is important to point out that the results must be given to first significant order in v/c or $\hbar q/mv$.

PART I: BASICS OF LASER COOLING

1. Absorption.

1a	Write down the resonance condition for the absorption of the photon. $\omega_0 \approx \omega_L\left(1 + \frac{v}{c}\right)$	0.2

1b	Write down the momentum p_{at} of the atom after absorption, as seen in the laboratory $p_{at} = p - \hbar q \approx mv - \frac{\hbar\omega_L}{c}$	0.2

1c	Write down the energy ε_{at} of the atom after absorption, as seen in the laboratory $\varepsilon_{at} = \frac{p_{at}^2}{2m} + \hbar\omega_0 \approx \frac{mv^2}{2} + \hbar\omega_L$	0.2

2. Spontaneous emission in the $-x$ direction.

First, one calculates the energy of the emitted photon, as seen in the lab reference frame. One must be careful to keep the correct order; this is because the velocity of the atom changes after the absorption, however, this is second order correction for the emitted frequency:

$$\omega_{ph} \approx \omega_0 \left(1 - \frac{v'}{c}\right) \quad \text{with} \quad v' \approx v - \frac{\hbar q}{m}$$

thus,

$$\begin{aligned}\omega_{ph} &\approx \omega_0 \left(1 - \frac{v}{c} + \frac{\hbar q}{mc}\right) \\ &\approx \omega_L \left(1 + \frac{v}{c}\right)\left(1 - \frac{v}{c} + \frac{\hbar q}{mc}\right) \\ &\approx \omega_L \left(1 + \frac{\hbar q}{mc}\right) \\ &\approx \omega_L \left(1 + \left(\frac{\hbar q}{mv}\right)\left(\frac{v}{c}\right)\right) \\ &\approx \omega_L\end{aligned}$$

2a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory. $\varepsilon_{ph} \approx \hbar\omega_L$	0.2

2b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory. $p_{ph} \approx -\hbar\omega_L / c$	0.2

Use conservation of momentum (see 1b):

$$p_{at} + p_{ph} \approx p - \hbar q$$

2c	Write down the momentum of the atom p_{at}, after the emission process in the $-x$ direction, as seen in the laboratory. $p_{at} \approx p = mv$	0.2

2d	Write down the energy of the atom ε_{at}, after the emission process in the $-x$ direction, as seen in the laboratory. $\varepsilon_{at} \approx \frac{p^2}{2m} = \frac{mv^2}{2}$	0.2

3. Spontaneous emission in the $+x$ direction.

The same as in the previous questions, keeping the right order

3a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory. $\varepsilon_{ph} \approx \hbar\omega_0\left(1+\frac{v}{c}\right) \approx \hbar\omega_L\left(1+\frac{v}{c}\right)\left(1+\frac{v}{c}\right) \approx \hbar\omega_L\left(1+2\frac{v}{c}\right)$	0.2

3b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory. $p_{ph} \approx \frac{\hbar\omega_L}{c}\left(1+2\frac{v}{c}\right)$	0.2

3c	Write down the momentum of the atom p_{at}, after the emission process in the $+x$ direction, as seen in the laboratory. $p_{at} = p - \hbar q - p_{ph} \approx p - \hbar q - \frac{\hbar\omega_L}{c}\left(1+2\frac{v}{c}\right) \approx mv - 2\frac{\hbar\omega_L}{c}$	0.2

3d	Write down the energy of the atom ε_{at}, after the emission process in the $+x$ direction, as seen in the laboratory. $\varepsilon_{at} = \frac{p_{at}^2}{2m} \approx \frac{mv^2}{2}\left(1-2\frac{\hbar q}{mv}\right)$	0.2

4. Average emission after absorption.

The spontaneous emission processes occur with equal probabilities in both directions.

4a	Write down the average energy of an emitted photon, ε_{ph}, after the emission process. $\varepsilon_{ph} = \frac{1}{2}\varepsilon_{ph}^{+} + \frac{1}{2}\varepsilon_{ph}^{-} \approx \hbar\omega_L\left(1+\frac{v}{c}\right)$	0.2

4b	Write down the average momentum of an emitted photon p_{ph}, after the emission process. $p_{ph} = \frac{1}{2}p_{ph}^{+} + \frac{1}{2}p_{ph}^{-} \approx \frac{\hbar\omega_L}{c}\frac{v}{c} = mv\left(\frac{\hbar q}{mv}\frac{v}{c}\right) \approx 0$ second order	0.2

4c	Write down the average energy of the atom ε_{at}, after the emission process. $\varepsilon_{at} = \frac{1}{2}\varepsilon_{at}^{+} + \frac{1}{2}\varepsilon_{at}^{-} \approx \frac{mv^2}{2}\left(1-\frac{\hbar q}{mv}\right)$	0.2

4d	Write down the average momentum of the atom p_{at}, after the emission process. $p_{at}=\frac{1}{2}p_{at}^{+}+\frac{1}{2}p_{at}^{-}\approx p-\frac{\hbar\omega_L}{c}$	0.2

5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser and the atom.

5a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. $\Delta\varepsilon=\varepsilon_{at}^{after}-\varepsilon_{at}^{before}\approx-\frac{1}{2}\hbar qv=-\frac{1}{2}\hbar\omega_L\frac{v}{c}$	0.2

5b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process. $\Delta p=p_{at}^{after}-p_{at}^{before}\approx-\hbar q=-\frac{\hbar\omega_L}{c}$	0.2

6. Energy and momentum transfer by a laser beam along the $+x$ direction.

6a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. $\Delta\varepsilon=\varepsilon_{at}^{after}-\varepsilon_{at}^{before}\approx+\frac{1}{2}\hbar qv=+\frac{1}{2}\hbar\omega_L'\frac{v}{c}$	0.3

6b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process. $\Delta p=p_{at}^{after}-p_{at}^{before}\approx+\hbar q=+\frac{\hbar\omega_L'}{c}$	0.3

PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Two counterpropagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a beam of N atoms that move in the $+x$ direction with (average) velocity v.

7. Force on the atomic beam by the lasers.

On the average, the fraction of atoms found in the excited state is given by,

$$P_{exc} = \frac{N_{exc}}{N} = \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where ω_0 is the resonance frequency of the atoms and Ω_R is the so-called Rabi frequency; Ω_R^2 is proportional to the *intensity* of the laser beam. The lifetime of the excited energy level of the atom is Γ^{-1}.

The force is calculated as the number of absorption-emission cycles, times the momentum exchange in each event, divided by the time of each event. CAREFUL! One must take into account the Doppler shift of each laser, as seen by the atoms:

7a	With the information found so far, find the force that the lasers exert on the atomic beam. You must assume that $mv >> \hbar q$. $F = N\Delta p^- P_{exc}^- \Gamma + N\Delta p^+ P_{exc}^+ \Gamma$ $= \left(\frac{\Omega_R^2}{\left(\omega_0 - \omega_L + \omega_L \frac{v}{c}\right)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2} - \frac{\Omega_R^2}{\left(\omega_0 - \omega_L - \omega_L \frac{v}{c}\right)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2} \right) N\Gamma\hbar q$	1.5

8. Low velocity limit.

Assume now the velocity to be small enough in order to expand the force to first order in v.

8a	Find an expression for the force found in Question (7a), in this limit. $F \approx -\frac{4N\hbar q^2 \Omega_R^2 \Gamma}{\left((\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2\right)^2} (\omega_0 - \omega_L) v$	1.5

8b	Write down the condition to obtain a positive force (speeding up the atom). $\omega_0 < \omega_L$	0.25

8c	Write down the condition to obtain a zero force. $\omega_0 = \omega_L$	0.25

8d	Write down the condition to obtain a negative force (slowing down the atom). $\omega_0 > \omega_L$ … this is the famous rule "tune below resonance for cooling down"	0.25

8e	Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms. $\omega_0 > \omega_L$ … i.e. independent of the direction motion of the atom.	0.25

9. Optical molasses

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, $t = 0$, the gas of atoms has velocity v_0.

9a	In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time τ. $F = -\beta v \Rightarrow m\frac{dv}{dt} \approx -\beta v$ β can be read from (8a) $\Rightarrow v = v_0 e^{-\beta t/m}$	1.5

9b	Assume now that the gas of atoms is in thermal equilibrium at a temperature T_0. Find the temperature T after the laser beams have been on for a time τ. Recalling that $\frac{1}{2}mv^2 = \frac{1}{2}kT$ in 1 dimension, and using v as the average thermal velocity in the equation of (9a), we can write down $T = T_0 e^{-2\beta t/m}$	0.5

Answers

Theoretical Problem No. 3

Why are stars so large?

1) A first, classic estimate of the temperature at the center of the stars.

1a	We equate the initial kinetic energy of the two protons to the electric potential energy at the distance of closest approach: $2(\frac{1}{2} m_p\, v_{rms}^2) = \frac{q^2}{4\pi\varepsilon_0 d_c}$; and since $\frac{3}{2} k T_c = \frac{1}{2} m_p\, v_{rms}^2$, we obtain $T_c = \frac{q^2}{12\pi\varepsilon_0 d_c k} = 5.5\times10^9\ \mathrm{K}$	1.5

2) Finding that the previous temperature estimate is wrong.

2a	Since we have that $\frac{\Delta P}{\Delta r} = -\frac{G M_r \rho_r}{r^2}$, making the assumptions given above, we obtain that: $P_c = \frac{G M \rho_c}{R}$. Now, the pressure of an ideal gas is $P_c = \frac{2\rho_c k T_c}{m_p}$, where k is Boltzmann´s constant, T_c is the central temperature of the star, and m_p is the proton mass. The factor of 2 in the previous equation appears because we have two particles (one proton and one electron) per proton mass and that both contribute equally to the pressure. Equating the two previous equations, we finally obtain that: $T_c = \frac{G M m_p}{2 k R}$	0.5

2b	From section (2a) we have that: $\frac{M}{R} = \frac{2 k T_c}{G m_p}$	0.5

2c	From section (2b) we have that, for $T_c = 5.5\times10^9$ K: $\frac{M}{R} = \frac{2kT_c}{Gm_p} = 1.4\times10^{24}$ kg m^{-1}.	0.5

2d	For the Sun we have that: $\frac{M(Sun)}{R(Sun)} = 2.9\times10^{21}$ kg m^{-1} , that is, three orders of magnitude smaller.	0.5

3) A quantum mechanical estimate of the temperature at the center of the stars

3a	We have that $\lambda_p = \frac{h}{m_p\, v_{rms}}$, and since $\frac{3}{2}kT_c = \frac{1}{2}m_p\, v_{rms}^2$, and $T_c = \frac{q^2}{12\pi\varepsilon_0 d_c k}$, we obtain: $T_c = \frac{q^4 m_p}{24\pi^2\varepsilon_0^2 k h^2}$.	1.0

3b	$T_c = \frac{q^4 m_p}{24\pi^2\varepsilon_0^2 k h^2} = 9.7\times10^6$ K.	0.5

3c	From section (2b) we have that, for $T_c = 9.7\times10^6$ K: $\frac{M}{R} = \frac{2kT_c}{Gm_p} = 2.4\times10^{21}$ kg m^{-1} ; while for the Sun we have that: $\frac{M(Sun)}{R(Sun)} = 2.9\times10^{21}$ kg m^{-1}.	0.5

4) The mass/radius ratio of the stars.

4a	Taking into account that	0.5

	$\frac{M}{R} = \frac{2kT_c}{G m_p}$, and that $T_c = \frac{q^4 m_p}{24\pi^2 \varepsilon_0^2 k h^2}$, we obtain: $\frac{M}{R} = \frac{q^4}{12\pi^2 \varepsilon_0^2 G h^2}$.	

5) The mass and radius of the smallest star.

5a	$n_e = \frac{M}{(4/3)\pi R^3 m_p}$	0.5

5b	$d_e = n_e^{-1/3} = \left(\frac{M}{(4/3)\pi R^3 m_p}\right)^{-1/3}$	0.5

5c	We assume that $d_e \geq \frac{\lambda_e}{2^{1/2}}$. Since $\lambda_e = \frac{h}{m_e v_{rms}(electron)}$, $\frac{3}{2}kT_c = \frac{1}{2}m_e v_{rms}^2(electron)$, $T_c = \frac{q^4 m_p}{24\pi^2 \varepsilon_0^2 k h^2}$, $\frac{M}{R} = \frac{q^4}{12\pi^2 \varepsilon_0^2 G h^2}$, and $d_e = \left(\frac{M}{(4/3)\pi R^3 m_p}\right)^{-1/3}$, we get that $R \geq \frac{\varepsilon_o^{1/2} h^2}{4^{1/4} q m_e^{3/4} m_p^{5/4} G^{1/2}}$	1.5

5d	$R \geq \frac{\varepsilon_o^{1/2} h^2}{4^{1/4} q m_e^{3/4} m_p^{5/4} G^{1/2}} = 6.9 \times 10^7$ m = 0.10 $R(Sun)$	0.5

5e	The mass to radius ratio is: $\frac{M}{R} = \frac{q^4}{12\pi^2 \varepsilon_0^2 G h^2} = 2.4\times10^{21}$ kg m^{-1}, from where we derive that $M \geq 1.7\times10^{29}$ kg = 0.09 $M(Sun)$	0.5

6) Fusing helium nuclei in older stars.

6a	For helium we have that $\frac{4q^2}{4\pi\varepsilon_0 m_{He} v_{rms}^2(He)} = \frac{h}{2^{1/2} m_{He} v_{rms}(He)}$; from where we get $v_{rms}(He) = \frac{2^{1/2} q^2}{\pi\varepsilon_0 h} = 2.0\times10^6$ m s^{-1}. We now use: $T(He) = \frac{v_{rms}^2(He) m_{He}}{3k} = 6.5\times10^8$ K. This value is of the order of magnitude of the estimates of stellar models.	0.5