1. Image of a charge in a metallic object

Introduction – Method of images

A point charge q is placed in the vicinity of a grounded metallic sphere of radius R [see Fig. 1(a)], and consequently a surface charge distribution is induced on the sphere. To calculate the electric field and potential from the distribution of the surface charge is a formidable task. However, the calculation can be considerably simplified by using the so called method of images. In this method, the electric field and potential produced by the charge distributed on the sphere can be represented as an electric field and potential of a single point charge q' placed inside the sphere (you do not have to prove it). Note: **The electric field of this image charge q' reproduces the electric field and the potential only outside the sphere (including its surface)**.

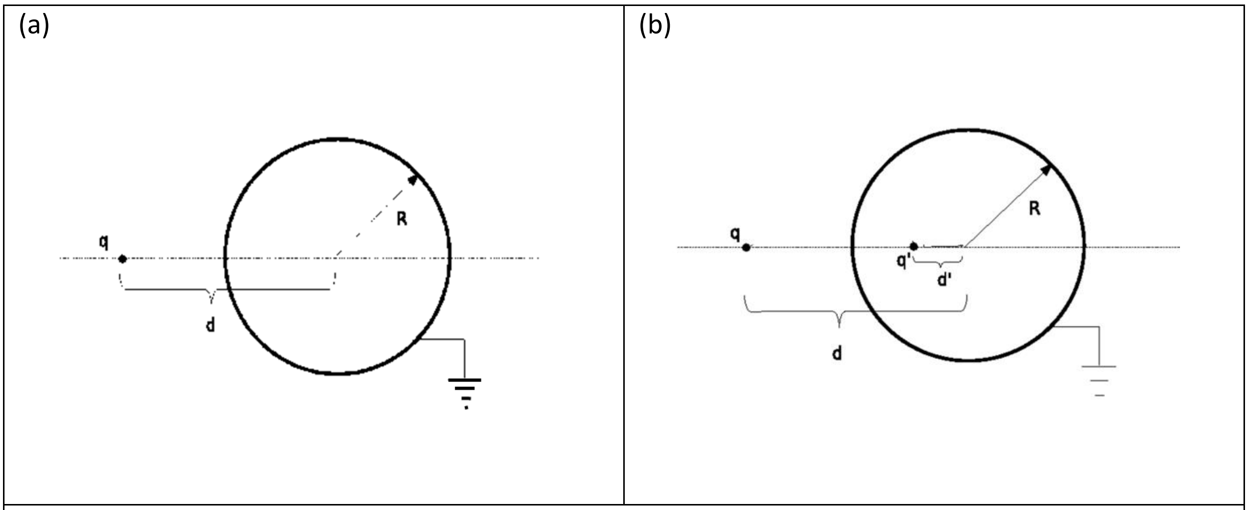

Fig 1. (a) A point charge q in the vicinity of a grounded metallic sphere. (b) The electric field of the charge induced on the sphere can be represented as electric field of an image charge q'.

Task 1 – The image charge

The symmetry of the problem dictates that the charge q' should be placed on the line connecting the point charge q and the center of the sphere [see Fig. 1(b)].

a) What is the value of the potential on the sphere? *(0.3 points)*

b) Express q' and the distance d' of the charge q' from the center of the sphere, in terms of q, d, and R. *(1.9 points)*

c) Find the magnitude of force acting on charge q. Is the force repulsive? *(0.5 points)*

Task 2 – Shielding of an electrostatic field

Consider a point charge *q* placed at a distance *d* from the center of a grounded metallic sphere of radius *R*. We are interested in how the grounded metallic sphere affects the electric field at point *A* on the opposite side of the sphere (see Fig. 2). Point *A* is on the line connecting charge *q* and the center of the sphere; its distance from the point charge *q* is *r*.

a) Find the vector of the electric field at point *A*. *(0.6 points)*

b) For a very large distance $r >> d$, find the expression for the electric field by using the approximation $(1+a)^{-2} \approx 1-2a$, where $a << 1$. *(0.6 points)*

c) In which limit of *d* does the grounded metallic sphere screen the field of the charge *q* completely, such that the electric field at point *A* is exactly zero? *(0.3 points)*

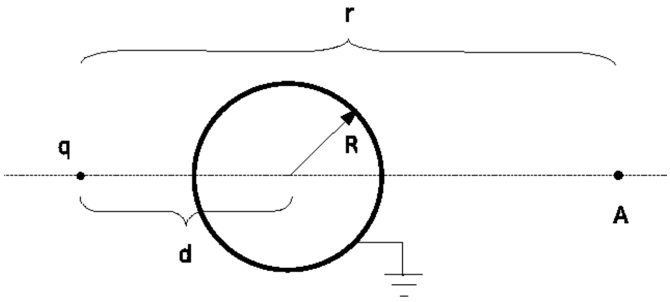

Fig 2. The electric field at point *A* is partially screened by the grounded sphere.

Task 3 – Small oscillations in the electric field of the grounded metallic sphere

A point charge *q* with mass *m* is suspended on a thread of length *L* which is attached to a wall, in the vicinity of the grounded metallic sphere. In your considerations, ignore all electrostatic effects of the wall. The point charge makes a mathematical pendulum (see Fig. 3). The point at which the thread is attached to the wall is at a distance l from the center of the sphere. Assume that the effects of gravity are negligible.

a) Find the magnitude of the electric force acting on the point charge *q* for a given angle α and indicate the direction in a clear diagram *(0.8 points)*

b) Determine the component of this force acting in the direction perpendicular to the thread in terms of l, *L*, *R*, *q and* α. *(0.8 points)*

c) Find the frequency for small oscillations of the pendulum. *(1.0 points)*

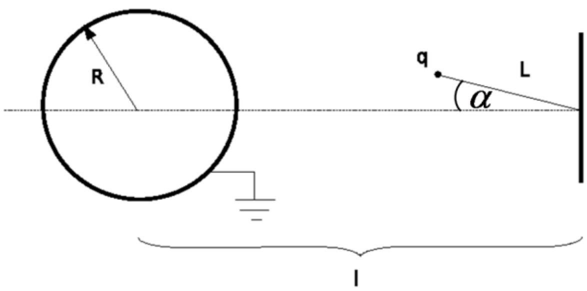

Fig 3. A point charge in the vicinity of a grounded sphere oscillates as a pendulum.

Task 4 – The electrostatic energy of the system

For a distribution of electric charges it is important to know the electrostatic energy of the system. In our problem (see Fig. 1a), there is an electrostatic interaction between the external charge *q* and the induced charges on the sphere, and there is an electrostatic interaction among the induced charges on the sphere themselves. In terms of the charge *q*, radius of the sphere *R* and the distance *d* determine the following electrostatic energies:

a) the electrostatic energy of the interaction between charge *q* and the induced charges on the sphere; *(1.0 points)*
b) the electrostatic energy of the interaction among the induced charges on the sphere; *(1.2 points)*
c) the total electrostatic energy of the interaction in the system. *(1.0 points)*

Hint: There are several ways of solving this problem:

(1) In one of them, you can use the following integral,

$$\int_d^\infty \frac{x\,dx}{\left(x^2 - R^2\right)^2} = \frac{1}{2}\frac{1}{d^2 - R^2}.$$

(2) In another one, you can use the fact that for a collection of *N* charges q_i located at points $\vec{r}_i, i = 1,\ldots,N$, the electrostatic energy is a sum over all pairs of charges: $V = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ i\neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{\left|\vec{r}_i - \vec{r}_j\right|}.$

2. Chimney physics

Introduction

Gaseous products of burning are released into the atmosphere of temperature T_{Air} through a high chimney of cross-section *A* and height *h* (see Fig. 1). The solid matter is burned in the furnace which is at temperature T_{Smoke}. The volume of gases produced per unit time in the furnace is *B*.

Assume that:

- the velocity of the gases in the furnace is negligibly small
- the density of the gases (smoke) does not differ from that of the air at the same temperature and pressure; while in furnace, the gases can be treated as ideal
- the pressure of the air changes with height in accordance with the hydrostatic law; the change of the density of the air with height is negligible
- the flow of gases fulfills the Bernoulli equation which states that the following quantity is conserved in all points of the flow:
  $\frac{1}{2}\rho v^2(z) + \rho g z + p(z) = const,$
  where ρ is the density of the gas, $v(z)$ is its velocity, $p(z)$ is pressure, and z is the height
- the change of the density of the gas is negligible throughout the chimney

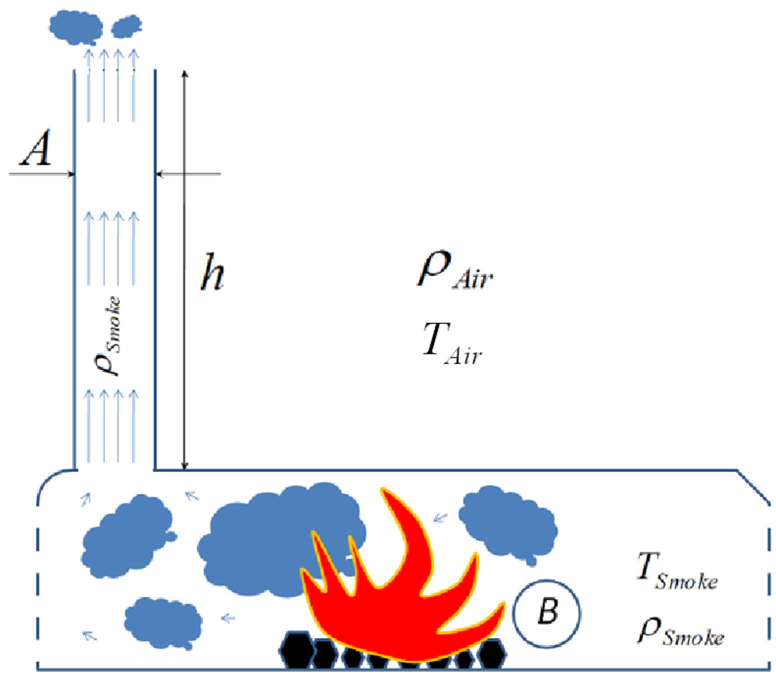

Fig 1. Sketch of a chimney of height *h* with a furnace at temperature T_{Smoke} .

Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas into the atmosphere? Express your result in terms of B, A, T_{Air}, g=9.81m/s^2, $\Delta T=T_{Smoke}-T_{Air}$. **Important: in all subsequent tasks assume that this minimal height is *the* height of the chimney.** *(3.5 points)*

b) Assume that two chimneys are built to serve exactly the same purpose. Their cross sections are identical, but are designed to work in different parts of the world: one in cold regions, designed to work at an average atmospheric temperature of -30 ^{o}C and the other in warm regions, designed to work at an average atmospheric temperature of 30 ^{o}C. The temperature of the furnace is 400 ^{o}C. It was calculated that the height of the chimney designed to work in cold regions is 100 m. How high is the other chimney? *(0.5 points)*

c) How does the velocity of the gases vary along the height of the chimney? Make a sketch/diagram assuming that the chimney cross-section does not change along the height. Indicate the point where the gases enter the chimney. *(0.6 points)*

d) How does the pressure of the gases vary along the height of the chimney? *(0.5 points)*

Solar power plant

The flow of gases in a chimney can be used to construct a particular kind of solar power plant (solar chimney). The idea is illustrated in Fig. 2. The Sun heats the air underneath the collector of area S with an open periphery to allow the undisturbed inflow of air (see Fig. 2). As the heated air rises through the chimney (thin solid arrows), new cold air enters the collector from its surrounding (thick dotted arrows) enabling a continuous flow of air through the power plant. The flow of air through the chimney powers a turbine, resulting in the production of electrical energy. The energy of solar radiation per unit time per unit of horizontal area of the collector is G. Assume that all that energy can be used to heat the air in the collector (the mass heat capacity of the air is c, and one can neglect its dependence on the air temperature). We define the efficiency of the solar chimney as the ratio of the kinetic energy of the gas flow and the solar energy absorbed in heating of the air prior to its entry into the chimney.

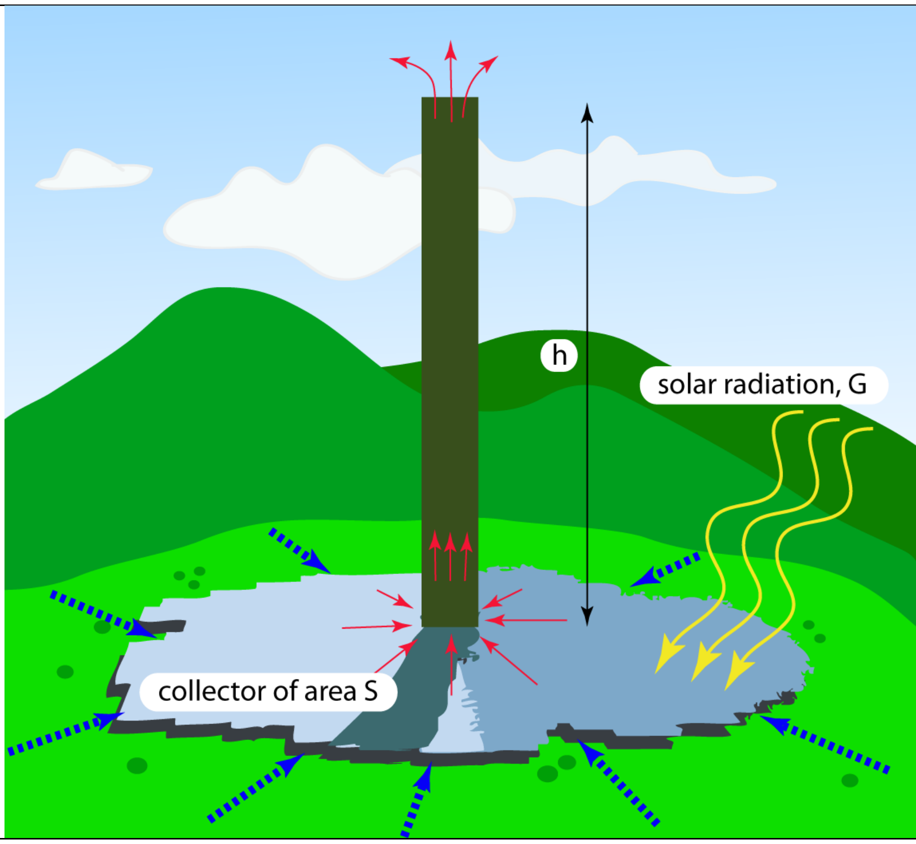

Fig 2. Sketch of a solar power plant.

Task 2

a) What is the efficiency of the solar chimney power plant? *(2.0 points)*

b) Make a diagram showing how the efficiency of the chimney changes with its height. *(0.4 points)*

Manzanares prototype

The prototype chimney built in Manzanares, Spain, had a height of 195 m, and a radius 5 m. The collector is circular with diameter of 244 m. The specific heat of the air under typical operational conditions of the prototype solar chimney is 1012 J/kg K, the density of the hot air is about 0.9 kg/m^3, and the typical temperature of the atmosphere T_{Air} = 295 K. In Manzanares, the solar power per unit of horizontal surface is typically 150 W/m^2 during a sunny day.

Task 3

a) What is the efficiency of the prototype power plant? Write down the numerical estimate. *(0.3 points)*

b) How much power could be produced in the prototype power plant? *(0.4 points)*

c) How much energy could the power plant produce during a typical sunny day? *(0.3 points)*

Task 4

a) How large is the rise in the air temperature as it enters the chimney (warm air) from the surrounding (cold air)? Write the general formula and evaluate it for the prototype chimney. *(1.0 points)*

b) What is the mass flow rate of air through the system? *(0.5 points)*

3. Simple model of an atomic nucleus

Introduction

Although atomic nuclei are quantum objects, a number of phenomenological laws for their basic properties (like radius or binding energy) can be deduced from simple assumptions: (i) nuclei are built from nucleons (i.e. protons and neutrons); (ii) strong nuclear interaction holding these nucleons together has a very short range (it acts only between neighboring nucleons); (iii) the number of protons (Z) in a given nucleus is approximately equal to the number of neutrons (N), i.e. $Z \approx N \approx A/2$, where A is the total number of nucleons ($A \gg 1$). **Important: Use these assumptions in Tasks 1-4 below.**

Task 1 - Atomic nucleus as closely packed system of nucleons

In a simple model, an atomic nucleus can be thought of as a ball consisting of closely packed nucleons [see Fig. 1(a)], where the nucleons are hard balls of radius $r_N = 0.85$ fm (1 fm = 10^{-15} m). The nuclear force is present only for two nucleons in contact. The volume of the nucleus V is larger than the volume of all nucleons AV_N, where $V_N = \frac{4}{3} r_N^3 \pi$. The ratio $f = AV_N / V$ is called the packing factor and gives the percentage of space filled by the nuclear matter.

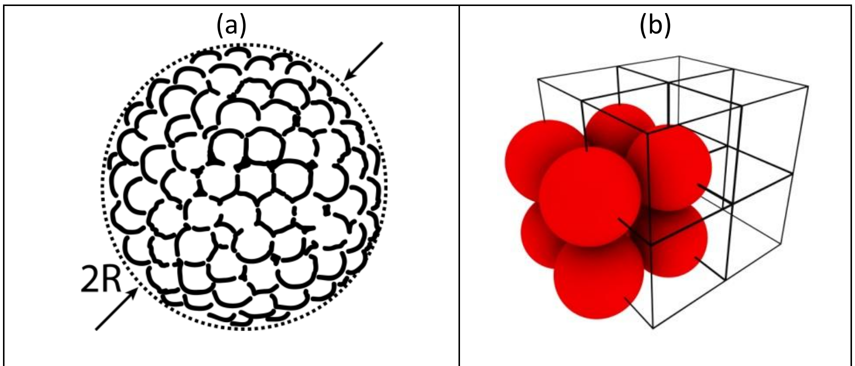

Fig. 1. (a) An atomic nucleus as a ball of closely packed nucleons. (b) The SC packing.

a) Calculate what would be the packing factor f if nucleons were arranged in a "simple cubic" (SC) crystal system, where each nucleon is centered on a lattice point of an infinite cubic lattice [see Fig. 1(b)]. *(0.3 points)*

Important: In all subsequent tasks, assume that the actual packing factor for nuclei is equal to the one from Task 1a. If you are not able to calculate it, in subsequent tasks use $f = 1/2$.

b) Estimate the average mass density ρ_m, charge density ρ_c, and the radius R for a nucleus having A nucleons. The average mass of a nucleon is $1.67 \cdot 10^{-27}$ kg. *(1.0 points)*

Task 2 - Binding energy of atomic nuclei – volume and surface terms

Binding energy of a nucleus is the energy required to disassemble it into separate nucleons and it essentially comes from the attractive nuclear force of each nucleon with its neighbors. If a given nucleon is not on the surface of the nucleus, it contributes to the total binding energy with a_V= 15.8 MeV (1 MeV = $1.602 \cdot 10^{-13}$ J). The contribution of one surface nucleon to the binding energy is approximately $a_V/2$. Express the binding energy E_b of a nucleus with A nucleons in terms of A, a_V, and f, and by including the surface correction. *(1.9 points)*

Task 3 - Electrostatic (Coulomb) effects on the binding energy

The electrostatic energy of a homogeneously charged ball (with radius R and total charge Q_0) is $U_c = \frac{3Q_0^2}{20\pi\varepsilon_0 R}$, where $\varepsilon_0 = 8.85 \cdot 10^{-12}\, \mathrm{C}^2 N^{-1} m^{-2}$.

a) Apply this formula to get the electrostatic energy of a nucleus. In a nucleus, each proton is not acting upon itself (by Coulomb force), but only upon the rest of the protons. One can take this into account by replacing $Z^2 \rightarrow Z(Z-1)$ in the obtained formula. Use this correction in subsequent tasks. *(0.4 points)*

b) Write down the complete formula for binding energy, including the main (volume) term, the surface correction term and the obtained electrostatic correction. *(0.3 points)*

Task 4 - Fission of heavy nuclei

Fission is a nuclear process in which a nucleus splits into smaller parts (lighter nuclei). Suppose that a nucleus with A nucleons splits into only two equal parts as depicted in Fig. 2.

a) Calculate the total kinetic energy of the fission products E_{kin} when the centers of two lighter nuclei are separated by the distance $d \geq 2R(A/2)$, where $R(A/2)$ is their radius. The large nucleus was initially at rest. *(1.3 points)*

b) Assume that $d = 2R(A/2)$ and evaluate the expression for E_{kin} obtained in part a) for A= 100, 150, 200 and 250 (express the results in units of MeV). Estimate the values of A for which fission is possible in the model described above? *(1.0 points)*

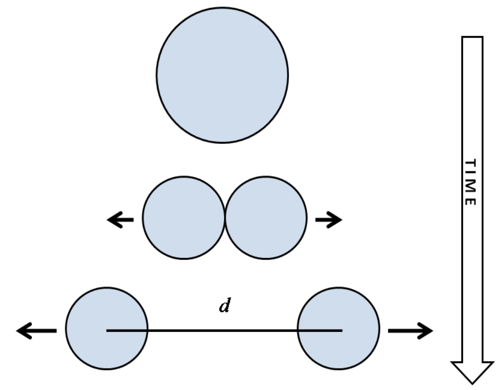

Fig. 2. A schematic description of nuclear fission in our model.

Task 5 – Transfer reactions

a) In modern physics, the energetics of nuclei and their reactions is described in terms of masses. For example, if a nucleus (with zero velocity) is in an excited state with energy E_{exc} above the ground state, its mass is $m = m_0 + E_{exc} / c^2$, where m_0 is its mass in the ground state at rest. The nuclear reaction ^{16}O+^{54}Fe→^{12}C+^{58}Ni is an example of the so-called "transfer reactions", in which a part of one nucleus ("cluster") is transferred to the other (see Fig. 3). In our example the transferred part is a ^{4}He-cluster (α -particle). The transfer reactions occur with maximum probability if the velocity of the projectile-like reaction product (in our case: ^{12}C) is equal both in magnitude and direction to the velocity of projectile (in our case: ^{16}O). The target ^{54}Fe is initially at rest. In the reaction, ^{58}Ni is excited into one of its higher-lying states. Find the excitation energy of that state (and express it units of MeV) if the kinetic energy of the projectile ^{16}O is 50 MeV. The speed of light is c= 3·10^{8} m/s. *(2.2 points)*

1.	M(^{16}O)	15.99491 a.m.u.
2.	M(^{54}Fe)	53.93962 a.m.u.
3.	M(^{12}C)	12.00000 a.m.u.
4.	M(^{58}Ni)	57.93535 a.m.u.

Table 1. The rest masses of the reactants in their ground states. 1 a.m.u.= 1.6605·10^{-27} kg.

b) The ^{58}Ni nucleus produced in the excited state discussed in the part a), deexcites into its ground state by emitting a gamma-photon in the direction of its motion. Consider this decay in the frame of reference in which ^{58}Ni is at rest to find the recoil energy of ^{58}Ni (*i.e.* kinetic energy which ^{58}Ni acquires after the emission of the photon). What is the photon energy in that system? What is the photon energy in the lab system of reference (i.e. what would be the energy of the photon measured in the detector which is positioned in the direction in which the ^{58}Ni nucleus moves)? *(1.6 points)*

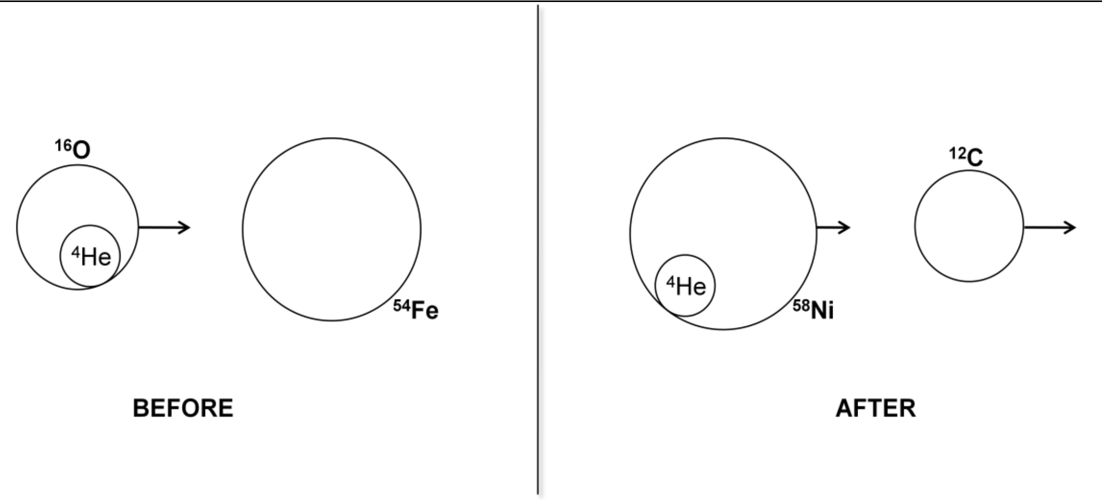

Fig. 3. The schematics of a transfer reaction.