1. A Three-body Problem and LISA

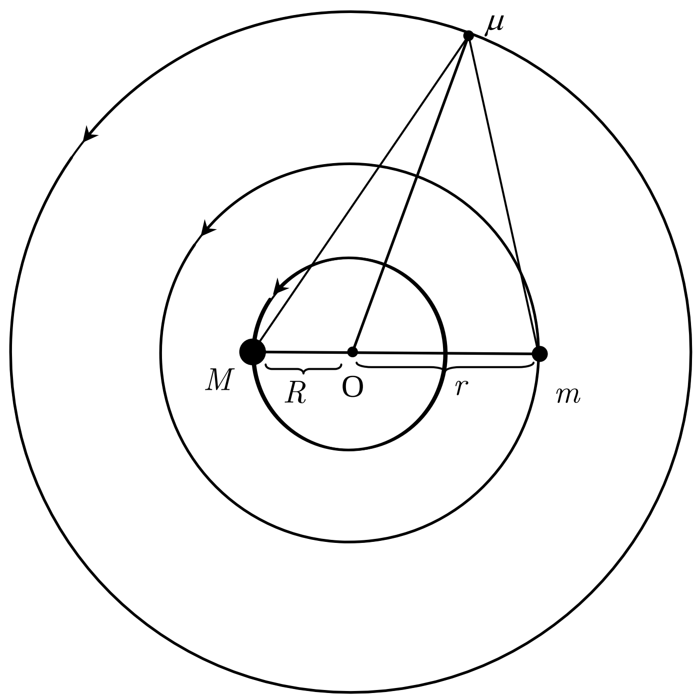

FIGURE 1 Coplanar orbits of three bodies.

1.1 Two gravitating masses M and m are moving in circular orbits of radii R and r, respectively, about their common centre of mass. Find the angular velocity ω_0 of the line joining M and m in terms of R, r, M, m and the universal gravitational constant G.

[1.5 points]

1.2 A third body of infinitesimal mass μ is placed in a coplanar circular orbit about the same centre of mass so that μ remains stationary relative to both M and m as shown in Figure 1. Assume that the infinitesimal mass is not collinear with M and m. Find the values of the following parameters in terms of R and r:

[3.5 points]

1.2.1 distance from μ to M.
1.2.2 distance from μ to m.
1.2.3 distance from μ to the centre of mass.

1.3 Consider the case $M = m$. If μ is now given a small radial perturbation (along Oμ), what is the angular frequency of oscillation of μ about the unperturbed position in terms of ω_0? Assume that the angular momentum of μ is conserved. **[3.2 points]**

The Laser Interferometry Space Antenna (LISA) is a group of three identical spacecrafts for detecting low frequency gravitational waves. Each of the spacecrafts is placed at the corners of an equilateral triangle as shown in Figure 2 and Figure 3. The sides (or 'arms') are about 5.0 million kilometres long. The LISA constellation is in an earth-like orbit around the Sun trailing the Earth by 20°. Each of them moves on a slightly inclined individual orbit around the Sun. Effectively, the three spacecrafts appear to roll about their common centre one revolution per year.

They are continuously transmitting and receiving laser signals between each other. Overall, they detect the gravitational waves by measuring tiny changes in the arm lengths using interferometric means. A collision of massive objects, such as blackholes, in nearby galaxies is an example of the sources of gravitational waves.

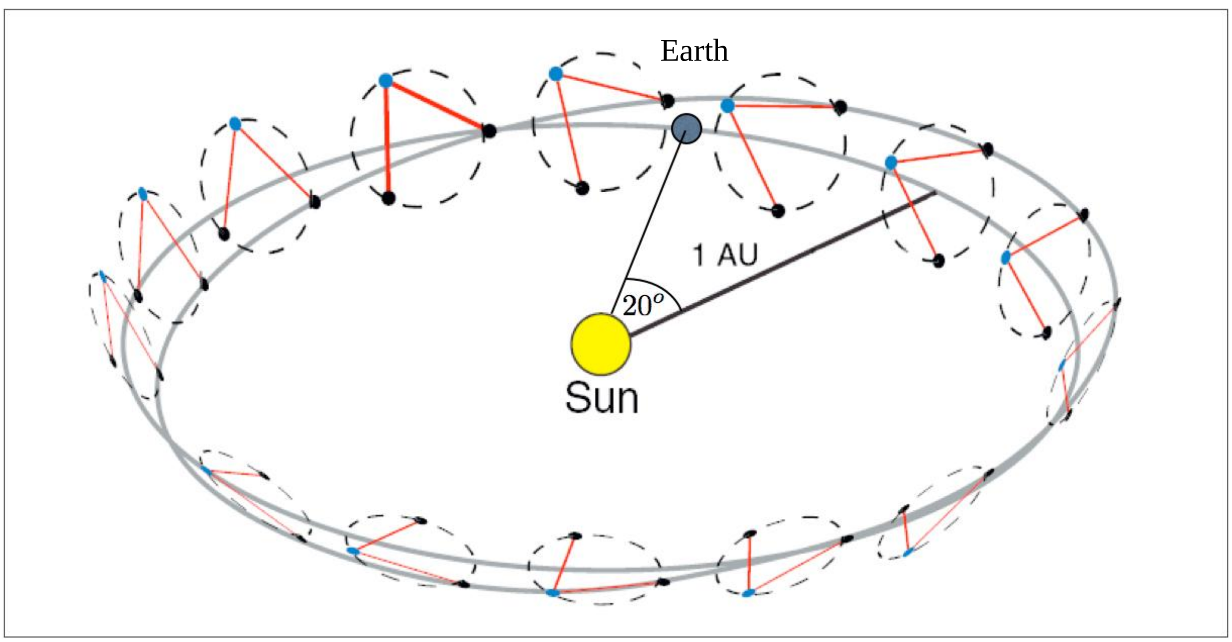

FIGURE 2 Illustration of the LISA orbit. The three spacecraft roll about their centre of mass with a period of 1 year. Initially, they trail the Earth by 20°. (Picture from D.A. Shaddock, "An Overview of the Laser Interferometer Space Antenna", *Publications of the Astronomical Society of Australia*, 2009, **26**, pp.128-132.).

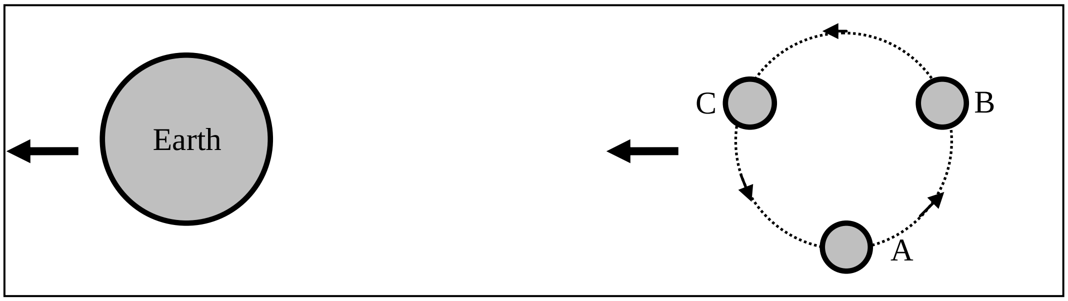

FIGURE 3 Enlarged view of the three spacecrafts trailing the Earth. A, B and C are the three spacecrafts at the corners of the equilateral triangle.

1.4 In the plane containing the three spacecrafts, what is the relative speed of one spacecraft with respect to another? **[1.8 point]**

2. An Electrified Soap Bubble

A spherical soap bubble with internal air density ρ_i, temperature T_i and radius R_0 is surrounded by air with density ρ_a, atmospheric pressure P_a and temperature T_a. The soap film has surface tension γ, density ρ_s and thickness t. The mass and the surface tension of the soap do not change with the temperature. Assume that $R_0 \gg t$.

The increase in energy, dE, that is needed to increase the surface area of a soap-air interface by dA, is given by $dE = \gamma dA$ where γ is the surface tension of the film.

2.1 Find the ratio $\frac{\rho_i T_i}{\rho_a T_a}$ in terms of γ, P_a and R_0. **[1.7 point]**

2.2 Find the numerical value of $\frac{\rho_i T_i}{\rho_a T_a} - 1$ using $\gamma = 0.0250\,\mathrm{Nm^{-1}}$, $R_0 = 1.00\ \mathrm{cm}$, and $P_a = 1.013 \times 10^5\ \mathrm{Nm^{-2}}$. **[0.4 point]**

2.3 The bubble is initially formed with warmer air inside. Find the minimum numerical value of T_i such that the bubble can float in still air. Use $T_a = 300\ \mathrm{K}$, $\rho_s = 1000\ \ \mathrm{kgm^{-3}}$, $\rho_a = 1.30\ \ \mathrm{kgm^{-3}}$, $t = 100\ \mathrm{nm}$ and $g = 9.80\ \ \mathrm{ms^{-2}}$. **[2.0 points]**

After the bubble is formed for a while, it will be in thermal equilibrium with the surrounding. This bubble in still air will naturally fall towards the ground.

2.4 Find the minimum velocity u of an updraught (air flowing upwards) that will keep the bubble from falling at thermal equilibrium. Give your answer in terms of ρ_s, R_0, g, t and the air's coefficient of viscosity η. You may assume that the velocity is small such that Stokes's law applies, and ignore the change in the radius when the temperature lowers to the equilibrium. The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$.

[1.6points]

2.5 Calculate the numerical value for u using $\eta = 1.8 \times 10^{-5}\ \mathrm{kgm^{-1}\,s^{-1}}$. **[0.4 point]**

The above calculations suggest that the terms involving the surface tension γ add very little to the accuracy of the result. In all of the questions below, you can neglect the surface tension terms.

2.6 If this spherical bubble is now electrified uniformly with a total charge q, find an equation describing the new radius R_1 in terms of R_0, P_a, q and the permittivity of free space ε_0.

[2.0points]

2.7 Assume that the total charge is not too large (i.e. $\frac{q^2}{\varepsilon_0 R_0^4} << P_a$) and the bubble only experiences a small increase in its radius, find ΔR where $R_1 = R_0 + \Delta R$.

Given that $(1+x)^n \approx 1 + nx$ where $x \ll 1$. **[0.7 point]**

2.8 What must be the magnitude of this charge q in terms of $t, \rho_a, \rho_s, \varepsilon_0, R_0, P_a$ in order that the bubble will float motionlessly in still air? Calculate also the numerical value of q. The permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ farad/m. **[1.2 point]**

3. To Commemorate the Centenary of Rutherford's Atomic Nucleus: the Scattering of an Ion by a Neutral Atom

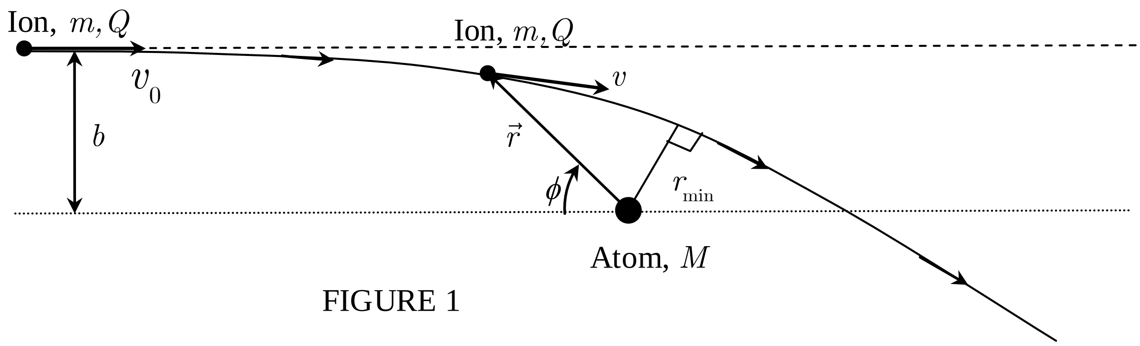

FIGURE 1

An ion of mass m, charge Q, is moving with an initial non-relativistic speed v_0 from a great distance towards the vicinity of a neutral atom of mass $M >> m$ and of electrical polarisability α. The impact parameter is b as shown in Figure 1.

The atom is instantaneously polarised by the electric field $\vec{E}$ of the in-coming (approaching) ion. The resulting electric dipole moment of the atom is $\vec{p} = \alpha \vec{E}$. Ignore any radiative losses in this problem.

3.1 Calculate the electric field intensity $\vec{E}_p$ at a distance r from an ideal electric dipole $\vec{p}$ at the origin O along the direction of $\vec{p}$ in Figure 2. **[1.2 points]**

$p = 2aq, \quad r \gg a$

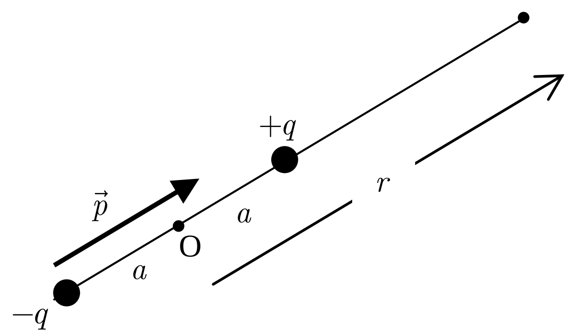

FIGURE 2

3.2 Find the expression for the force $\vec{f}$ acting on the ion due to the polarised atom. Show that this force is attractive regardless of the sign of the charge of the ion.

[3.0 points]

3.3 What is the electric potential energy of the ion-atom interaction in terms of α, Q and r?

[0.9 points]

3.4 Find the expression for $r_{\min}$, the distance of the closest approach, as shown in Figure 1.

[2.4 points]

3.5 If the impact parameter b is less than a critical value b_0, the ion will descend along a spiral to the atom. In such a case, the ion will be neutralized, and the atom is, in turn, charged. This process is known as the "charge exchange" interaction. What is the cross sectional area $A = \pi b_0^2$ of this "charge exchange" collision of the atom as seen by the ion? **[2.5 points]**