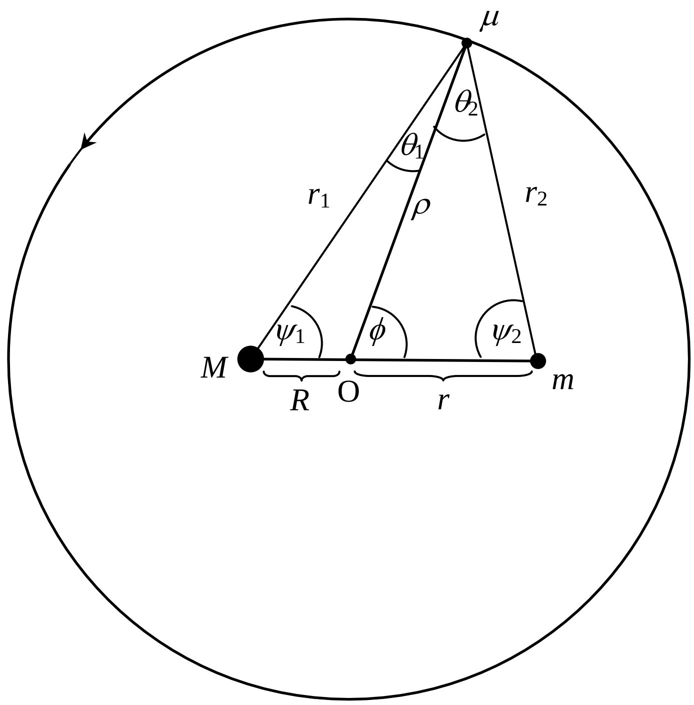
I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \qquad \text{(1)}$$

$$m\omega_0^2 r = \frac{GMm}{(R+r)^2}$$

$$M\omega_0^2 R = \frac{GMm}{(R+r)^2} \qquad \text{(2)}$$

From Eq. (2), or using reduced mass, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$

Hence, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3} = \frac{GM}{r(R+r)^2} = \frac{Gm}{R(R+r)^2}$. ……………………………… (3)

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m. For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.......................... (4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.......................... (5)}$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \qquad \text{.......................... (6)}$$

The distances r_2 and ρ, the angles θ_1 and θ_2 are related by two Sine Rule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$
$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r} \qquad \text{.......................... (7)}$$

Substitute (7) into (6)

$$\frac{1}{r_2^3} = \frac{R}{(R+r)^4}\frac{(M+m)}{m} \qquad \text{.......................... (10)}$$

Since $\frac{m}{M+m} = \frac{R}{R+r}$, Eq. (10) gives

$$r_2 = R + r \qquad \text{.......................... (11)}$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R + r \qquad \text{.......................... (12)}$$

Alternatively, $\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$ and $\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\psi_1 = 60^\circ \qquad \psi_2 = 60^\circ \qquad \text{.......................... (13)}$$

The distance ρ is calculated from the Cosine Rule.

$$\rho^2 = r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ \qquad \rho = \sqrt{r^2 + rR + R^2} \qquad \text{.......................... (14)}$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.......................... (4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.......................... (5)}$$

Note that $\dfrac{r_1}{\sin(180^\circ - \phi)} = \dfrac{R}{\sin\theta_1}$

$\dfrac{r_2}{\sin\phi} = \dfrac{r}{\sin\theta_2}$ (see figure)

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1} \qquad \text{.......................... (6)}$$

Equations (5) and (6):

$$r_1 = r_2 \qquad \text{.......................... (7)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \qquad \text{.......................... (8)}$$

$$\psi_1 = \psi_2 \qquad \text{.......................... (9)}$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3}r_1^2\rho \qquad \text{.......................... (10)}$$

Equations (8) and (10): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2\rho}{(R+r)^3}\sin\theta_2$ .... (11)

Note that from figure, $\dfrac{\rho}{\sin\psi_2} = \dfrac{r}{\sin\theta_2}$ .... (12)

Equations (11) and (12): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2 r}{(R+r)^3}\sin\psi_2$ ………………………. (13)

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2)+r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \quad \text{……………… (14)}$$

Equations (13) and (14): $\sin(\theta_1+\theta_2) = \dfrac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]}$ ………………………. (15)

$$\theta_1+\theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \quad \text{(see figure)}$$

$$\therefore \quad \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is $\rho = \sqrt{\left(\dfrac{R+r}{2}-R\right)^2+\left\{(R+r)\dfrac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$

1.3 The energy of the mass μ is given by

$$E = -\frac{GM\mu}{r_1} - \frac{Gm\mu}{r_2} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2+\rho^2\omega^2\right) \quad \text{………………………..(15)}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1 = r_2 = \Re$ and $m = M$),

$$E = -\frac{2GM\mu}{\Re} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{\rho_0^4\omega_0^2}{\rho^2}\right) \quad \text{………………………..(16)}$$

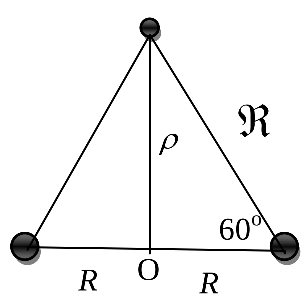
Since the energy is conserved,

$$\frac{dE}{dt} = 0$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^2}\frac{d\Re}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(17)}$$

$$\frac{d\Re}{dt} = \frac{d\Re}{d\rho}\frac{d\rho}{dt} = \frac{d\rho}{dt}\frac{\rho}{\Re} \quad \text{……………(18)}$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^3}\rho\frac{d\rho}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(19)}$$



Since $\frac{d\rho}{dt} \neq 0$, we have

$$\frac{2GM}{\Re^3}\rho + \frac{d^2\rho}{dt^2} - \frac{\rho_0{}^4\omega_0{}^2}{\rho^3} = 0 \text{ or}$$

$$\frac{d^2\rho}{dt^2} = -\frac{2GM}{\Re^3}\rho + \frac{\rho_0{}^4\omega_0{}^2}{\rho^3}. \qquad \text{.............................(20)}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re = \Re_0\left(1+\frac{\Delta\Re}{\Re_0}\right)$ and $\rho = \rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dt^2}(\rho_0 + \Delta\rho) = -\frac{2GM}{\Re_0^3\left(1+\frac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right) + \frac{\rho_0{}^4\omega_0{}^2}{\rho_0^3\left(1+\frac{\Delta\rho}{\rho_0}\right)^3} \qquad \text{..................(21)}$$

Using binomial expansion $(1+\varepsilon)^n \approx 1+n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)\left(1-\frac{3\Delta\Re}{\Re_0}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{..................(22)}$$

Using $\Delta\rho = \frac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{..................(23)}$$

Since $\omega_0^2 = \frac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \omega_0^2\rho_0\left(1-\frac{3\Delta\rho}{\rho_0}\right) \qquad \text{..................(24)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \qquad \text{..................(25)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \qquad \text{..................(26)}$$

From the figure, $\rho_0 = \Re_0\cos 30°$ or $\frac{\rho_0{}^2}{\Re_0{}^2} = \frac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4-\frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho\,. \qquad \text{..................(27)}$$

Angular frequency of oscillation is $\frac{\sqrt{7}}{2}\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^2 = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R + \zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2nd law, $-\frac{2GM\mu}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R+\zeta) - \mu\omega^2(\sqrt{3}R+\zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R+\zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2 + (\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2 + 2\sqrt{3}\zeta R\}^{3/2}}(\sqrt{3}R+\zeta) \approx \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}} = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\omega_0^2\sqrt{3}R\left(1-\frac{3\sqrt{3}\zeta}{4R}\right)\left(1+\frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2\zeta}{dt^2} - \omega_0^2\sqrt{3}R\left(1-\frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta = -\left(\frac{7}{4}\omega_0^2\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, v_{BA} is the velocity of B as observed by A.

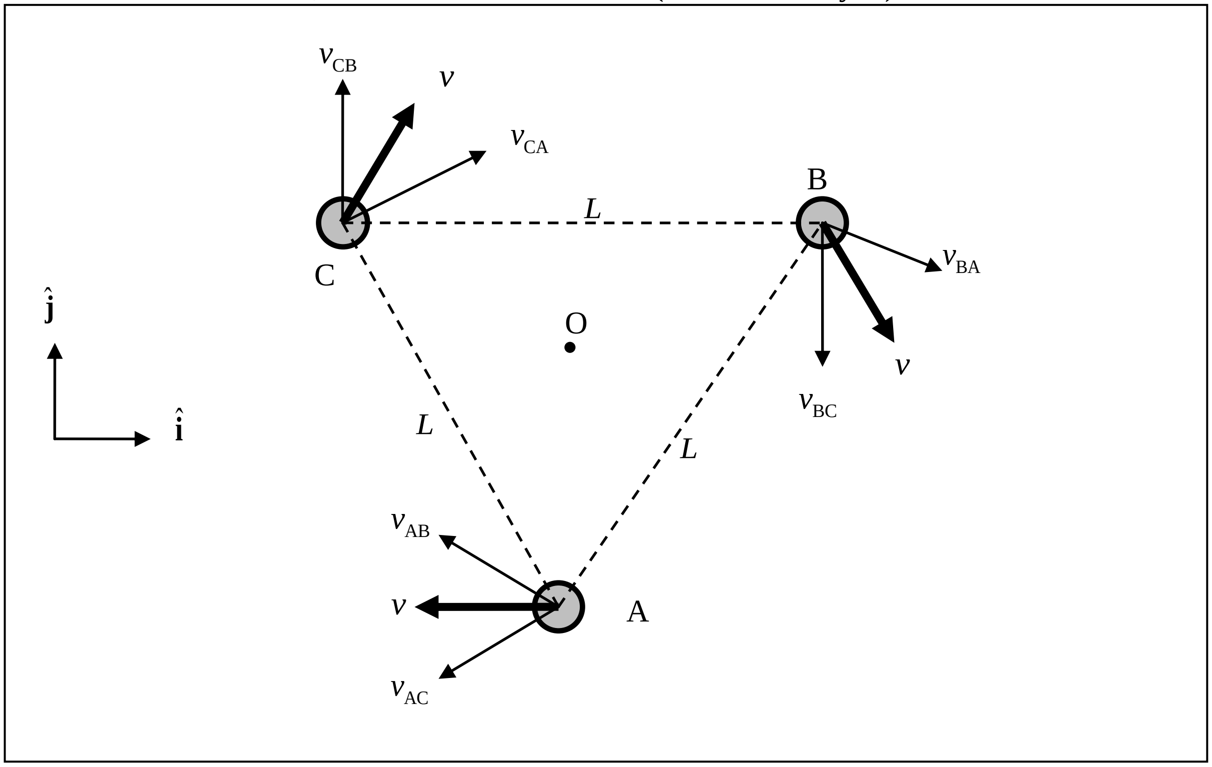
The period of circular motion is 1 year $T = 365\times 24\times 60\times 60$ s. .... (28)

The angular frequency $\omega = \frac{2\pi}{T}$

The speed $v = \omega\frac{L}{2\cos 30°} = 575$ m/s .... (29)

The speed is much less than the speed light → Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60°\hat{\mathbf{i}} - v\sin 60°\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60°\hat{\mathbf{i}} + v\sin 60°\hat{\mathbf{j}}$

Hence $\vec{v}_{\text{BC}} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s ………… (30)

Notice that the relative velocities for each pair are anti-parallel.

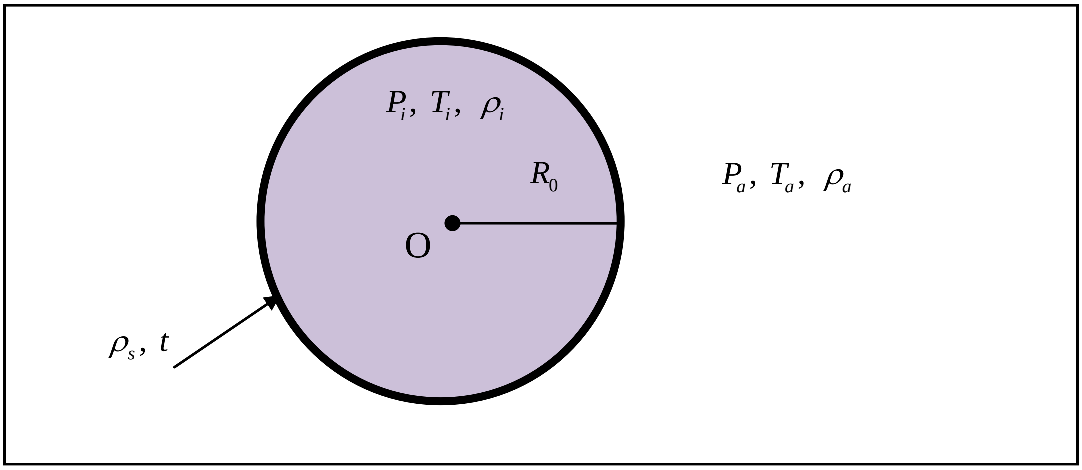
Alternative solution for 1.4

One can obtain v_{BC} by considering the rotation about the axis at one of the spacecrafts.

$$v_{\text{BC}} = \omega L = \frac{2\pi}{365\times 24\times 60\times 60\ \text{s}}(5\times 10^6\ \text{km}) \approx 996\ \text{m/s}$$

2. SOLUTION

2.1. The bubble is surrounded by air.



Cutting the sphere in half and using the projected area to balance the forces give

$$P_i \pi R_0^2 = P_a \pi R_0^2 + 2\left(2\pi R_0 \gamma\right)$$
$$P_i = P_a + \frac{4\gamma}{R_0} \qquad \ldots (1)$$

The pressure and density are related by the ideal gas law:

$PV = nRT$ or $P = \dfrac{\rho RT}{M}$, where M = the molar mass of air. … (2)

Apply the ideal gas law to the air inside and outside the bubble, we get

$$\rho_i T_i = P_i \frac{M}{R}$$
$$\rho_a T_a = P_a \frac{M}{R},$$

$$\frac{\rho_i T_i}{\rho_a T_a} = \frac{P_i}{P_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \ldots (3)$$

2.2. Using $\gamma = 0.025\,\text{Nm}^{-1}$, $R_0 = 1.0$ cm and $P_a = 1.013\times10^5\,\text{Nm}^{-2}$, the numerical value of the ratio is

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{R_0 P_a} = 1 + 0.0001 \qquad \dots (4)$$

(The effect of the surface tension is very small.)

2.3. Let W = total weight of the bubble, F = buoyant force due to air around the bubble

$$W = (\text{mass of film+mass of air})g$$
$$= \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right)g \qquad \dots (5)$$
$$= 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right]g$$

The buoyant force due to air around the bubble is

$$B = \frac{4}{3}\pi R_0^3 \rho_a g \qquad \dots (6)$$

If the bubble floats in still air,

$$B \ge W$$
$$\frac{4}{3}\pi R_0^3 \rho_a g \ge 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right]g \qquad \dots (7)$$

Rearranging to give

$$T_i \ge \frac{R_0 \rho_a T_a}{R_0 \rho_a - 3\rho_s t}\left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \dots (8)$$
$$\ge 307.1\ \text{K}$$

The air inside must be about 7.1°C warmer.

2.4. Ignore the radius change → Radius remains $R_0 = 1.0 \text{ cm}$

(The radius actually decreases by 0.8% when the temperature decreases from 307.1 K to 300 K. The film itself also becomes slightly thicker.)

The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$ … (9)

If the bubble floats in the updraught,

$$F \geq W - B$$

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g - \frac{4}{3}\pi R_0^3 \rho_a g \qquad \text{… (10)}$$

When the bubble is in thermal equilibrium $T_i = T_a$.

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_a \left[1 + \frac{4\gamma}{R_0 P_a}\right]\right) g - \frac{4}{3}\pi R_0^3 \rho_a g$$

Rearranging to give

$$u \geq \frac{4R_0\,\rho_s t g}{6\eta} + \frac{\frac{4}{3}R_0^2 \rho_a g \left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta} \qquad \text{… (11)}$$

2.5. The numerical value is $u \geq 0.36 \text{ m/s}$.

The 2nd term is about 3 orders of magnitude lower than the 1st term.

From now on, ignore the surface tension terms.

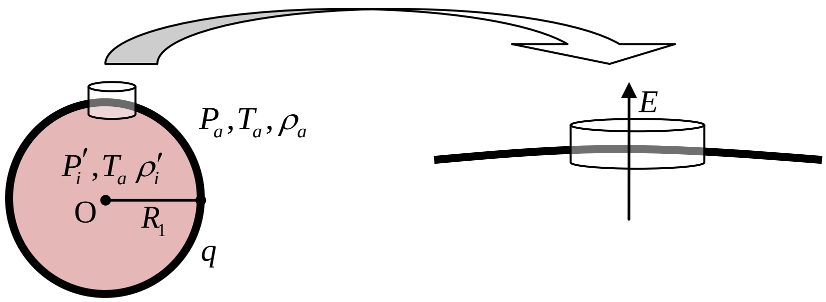
2.6. When the bubble is electrified, the electrical repulsion will cause the bubble to expand in size and thereby raise the buoyant force.

The force/area is (e-field on the surface × charge/area)

There are two alternatives to calculate the electric field ON the surface of the soap film.

A. From Gauss's Law

Consider a very thin pill box on the soap surface.



E = electric field on the film surface that results from all other parts of the soap film, excluding the surface inside the pill box itself.

E_q = total field just outside the pill box = $\dfrac{q}{4\pi\varepsilon_0 R_1^2} = \dfrac{\sigma}{\varepsilon_0}$

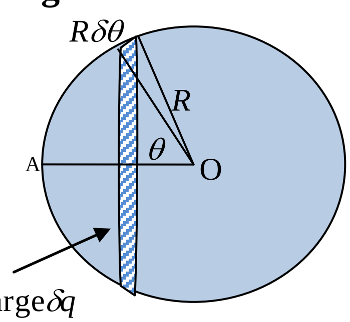
= E + electric field from surface charge σ

= $E + E_\sigma$

Using Gauss's Law on the pill box, we have $E_\sigma = \dfrac{\sigma}{2\varepsilon_0}$ perpendicular to the film as a result of symmetry.

Therefore, $E = E_q - E_\sigma = \dfrac{\sigma}{\varepsilon_0} - \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\sigma}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}\dfrac{q}{4\pi R_1^2}$ … (12)

B. From direct integration



$$\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\delta\theta$$

To find the magnitude of the electrical repulsion we must first find the electric field intensity E at a point <u>on</u> (not outside) the surface itself.

Field at A in the direction $\overrightarrow{OA}$ is

$$\delta E_A = \frac{1}{4\pi\varepsilon_0}\frac{\left(q/4\pi R_1^2\right)2\pi R_1^2\sin\theta\delta\theta}{\left(2R_1\sin\frac{\theta}{2}\right)^2}\sin\frac{\theta}{2} = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\cos\frac{\theta}{2}\delta\left(\frac{\theta}{2}\right)$$

$$E_A = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\int_{\theta=0}^{\theta=180^\circ}\cos\frac{\theta}{2}d\left(\frac{\theta}{2}\right) = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \quad \ldots (13)$$

The repulsive force per unit area of the surface of bubble is

$$\left(\frac{q}{4\pi R_1^2}\right)E = \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} \quad \ldots (14)$$

Let P_i' and ρ_i' be the new pressure and density when the bubble is electrified.

This electric repulsive force will augment the gaseous pressure P_i'.

P_i' is related to the original P_i through the gas law.

$$P_i'\frac{4}{3}\pi R_1^3 = P_i\frac{4}{3}\pi R_0^3$$

$$P_i' = \left(\frac{R_0}{R_1}\right)^3 P_i = \left(\frac{R_0}{R_1}\right)^3 P_a \quad \ldots (15)$$

<u>In the last equation, the surface tension term has been ignored.</u>

From balancing the forces on the half-sphere projected area, we have (again ignoring the surface tension term)

$$P_i' + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a$$

$$P_a\left(\frac{R_0}{R_1}\right)^3 + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a \quad \ldots (16)$$

Rearranging to get

$$\left(\frac{R_1}{R_0}\right)^4 - \left(\frac{R_1}{R_0}\right) - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0 \qquad \dots (17)$$

Note that (17) yields $\frac{R_1}{R_0} = 1$ when $q = 0$, as expected.

2.7. Approximate solution for R_1 when $\frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} << 1$

Write $R_1 = R_0 + \Delta R,\ \Delta R << R_0$

Therefore, $\frac{R_1}{R_0} = 1 + \frac{\Delta R}{R_0},\ \left(\frac{R_1}{R_0}\right)^4 \approx 1 + 4\frac{\Delta R}{R_0}$ … (18)

Eq. (17) gives:

$$\Delta R \approx \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \qquad \dots (19)$$

$$R_1 \approx R_0 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right) \qquad \dots (20)$$

2.8. The bubble will float if

$$B \geq W$$

$$\frac{4}{3}\pi R_1^3 \rho_a g \geq 4\pi R_0^2\, \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_i g \qquad \dots (21)$$

Initially, $T_i = T_a \Rightarrow \rho_i = \rho_a$ for $\gamma \to 0$ and $R_1 = R_0\left(1 + \frac{\Delta R}{R_0}\right)$

$$\frac{4}{3}\pi R_0^3\left(1+\frac{\Delta R}{R_0}\right)^3 \rho_a g \geq 4\pi R_0^2\,\rho_s t g+\frac{4}{3}\pi R_0^3\rho_a g$$

$$\frac{4}{3}\pi\left(3\Delta R\right)\rho_a g \geq 4\pi R_0^2\,\rho_s t g$$

$$\frac{4}{3}\pi\frac{3q^2}{96\pi^2\varepsilon_0 R_0 P_a}\rho_a g \geq 4\pi R_0^2\,\rho_s t g \qquad \ldots (22)$$

$$q^2 \geq \frac{96\pi^2 R_0^3\,\rho_s t\varepsilon_0 P_a}{\rho_a}$$

$$q \approx 256\times10^{-9}\ \text{C} \approx 256\,\text{nC}$$

Note that if the surface tension term is retained, we get

$$R_1 \approx \left(1+\frac{q^2/96\pi^2\varepsilon_0 R_0^4 P_a}{\left[1+\frac{2}{3}\left(\frac{4\gamma}{R_0 P_a}\right)\right]}\right)R_0$$

QUESTION 3: SOLUTION

1. Using Coulomb's Law, we write the electric field at a distance r is given by

$$E_p = \frac{q}{4\pi\varepsilon_0(r-a)^2} - \frac{q}{4\pi\varepsilon_0(r+a)^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Using binomial expansion for small a,

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r}-1+\frac{2a}{r}\right)$$

$$= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

$$= \frac{2p}{4\pi\varepsilon_0 r^3}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

The induced dipole moment is then simply

$$\vec{p} = \alpha\vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \qquad \ldots\ldots\ldots\ldots\ldots (5)$$

The "-'' sign implies that this force is attractive and Q^2 implies that the force is attractive regardless of the sign of Q.

3. The potential energy of the ion-atom is given by $U=\int_r^{\infty}\vec{f}.d\vec{r}$ ……….……………………….(6)

Using this, $U=\int_r^{\infty}\vec{f}.d\vec{r}=-\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ ……………………………………………………….(7)

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$mv_{\max}r_{\min} = mv_0 b$$

$$v_{\max} = v_0\frac{b}{r_{\min}} \qquad \text{…………….. (8)}$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2}mv_{\max}^2+\frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \frac{1}{2}mv_0^2 \qquad \text{…………….. (9)}$$

Eqs.(12) & (13):

$$\left(\frac{b}{r_{\min}}\right)^2-\frac{\alpha Q^2\Big/\frac{1}{2}mv_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4-\left(\frac{r_{\min}}{b}\right)^2+\frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 mv_0^2 b^4} = 0 \qquad \text{…………….. (10)}$$

The roots of eq. (14) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1\pm\sqrt{1-\frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………….. (11)}$$

[Note that the equation (14) implies that $r_{\min}$ cannot be zero, unless b is itself zero.]

Since the expression has to be valid at $Q=0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}[1\pm 1]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min}=b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1+\sqrt{1-\frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………………………………….....(12)}$$

5. A spiral trajectory occurs when (16) is imaginary (because there is no minimum distance of approach).

$r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}} \quad \text{……………..} (13)$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area A,

$$A = \pi b_0^2 = \pi \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{2}} \quad \text{……………..} (14)$$