Data sheet: Table of physical parameters

Speed of light in vacuum	$c = 2.998 \times 10^{8}$ m s^{-1}
Planck's constant over 2π	$\hbar = 1.055 \times 10^{-34}$ J s
Gravitational constant	$G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}
Gravitational acceleration	$g = 9.82$ m s^{-2}
Elementary charge	$e = 1.602 \times 10^{-19}$ C
Electric permittivity of vacuum	$\varepsilon_0 = 8.854 \times 10^{-12}$ C^2 J^{-1} m^{-1}
Electron mass	$m_e = 9.109 \times 10^{-31}$ kg
Avogadro constant	$N_A = 6.022 \times 10^{23}$ mol^{-1}
Boltzmann constant	$k_B = 1.381 \times 10^{-23}$ J K^{-1}
Stony meteorite, specific heat	$c_{sm} = 1.2 \times 10^{3}$ J kg^{-1}K^{-1}
Stony meteorite, thermal conductivity	$k_{sm} = 2.0$ W m^{-1} K^{-1}
Stony meteorite, density	$\rho_{sm} = 3.3 \times 10^{3}$ kg m^{-3}
Stony meteorite, melting point	$T_{sm} = 1.7 \times 10^{3}$ K
Stony meteorite, specific melting heat	$L_{sm} = 2.6 \times 10^{5}$ J kg^{-1}
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Sun, radius of the	$R_S = 6.96 \times 10^{8}$ m
Average Sun-Earth distance	$a_E = 1.50 \times 10^{11}$ m

Introduction

A meteoroid is a small particle (typically smaller than 1 m) from a comet or an asteroid. A meteoroid that impacts the ground is called a meteorite.

On the night of 17 January 2009 many people near the Baltic Sea saw the glowing trail or fireball of a meteoroid falling through the atmosphere of the Earth. In Sweden a surveillance camera recorded a video of the event, see Fig. 1.1(a). From these pictures and eyewitness accounts it was possible to narrow down the impact area, and six weeks later a meteorite with the mass 0.025 kg was found in the vicinity of the town Maribo in southern Denmark. Measurements on the meteorite, now named Maribo, and its orbit in the sky showed interesting results. Its speed when entering the atmosphere had been exceptionally high. Its age, 4.567×10^9 year, shows that it had been formed shortly after the birth of the solar system. The Maribo meteorite is possibly a part of Comet Encke.

The speed of Maribo

The fireball was moving in westerly direction, heading 285° relative to north, toward the location where the meteorite was subsequently found, as sketched in Fig. 1.1. The meteorite was found at a distance 195 km from the surveillance camera in the direction 230° relative to north.

1.1	Use this and the data in Fig. 1.1 to calculate the average speed of Maribo in the time interval between frames 155 and 161. The curvature of the Earth and the gravitational force on the meteoroid can both be neglected.	1.3

Through the atmosphere and melting?

The friction from the air on a meteoroid moving in the higher atmosphere depends in a complicated way on the shape and velocity of the meteoroid, and on the temperature and density of the atmosphere. As a reasonable approximation the friction force F in the upper atmosphere is given by the expression $F = k\rho_{\mathrm{atm}}Av^2$, where k is a constant, ρ_{atm} the density of the atmosphere, A the projected cross-section area of the meteorite, and v its speed.

The following simplifying assumptions are made to analyze the meteoroid: The object entering the atmosphere was a sphere of mass $m_{\mathrm{M}} = 30$ kg, radius $R_{\mathrm{M}} = 0.13$ m, temperature $T_0 = 200$ K, and speed $v_{\mathrm{M}} = 2.91 \times 10^4$ m/s. The density of the atmosphere is constant (its value 40 km above the surface of the Earth), $\rho_{\mathrm{atm}} = 4.1 \times 10^{-3}$ kg/m^3, and the friction coefficient is $k = 0.60$.

1.2a	Estimate how long time after entering the atmosphere it takes the meteoroid to have its speed reduced by 10 % from v_{M} to 0.90 v_{M}. You can neglect the gravitational force on the meteoroid and assume, that it maintains its mass and shape.	0.7
1.2b	Calculate how many times larger the kinetic energy E_{kin} of the meteoroid entering the atmosphere is than the energy E_{melt} necessary for melting it completely (see data sheet).	0.3

(a)

(b)

Frame	Time	Azimuth	Altitude
155	1.46 s	215°	19.2°
161	2.28 s	221°	14.7°
Landing at M		230°	0.0°

(c)

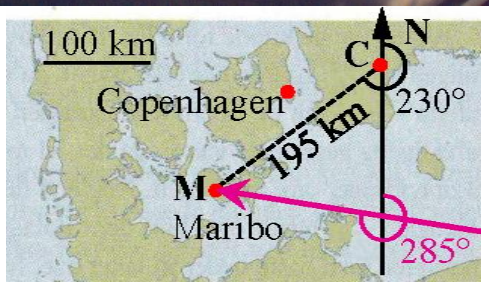

Figure 1.1 (a) Azimuth is the clockwise angular position from north in the horizontal plane, and altitude is the angular position above the horizon. A series of frames recorded by the surveillance camera in Sweden, showing the motion of Maribo as a fireball on its way down through the atmosphere. **(b)** The data from two frames indicating the time, the direction (azimuth) in degrees, as seen by the camera (C), and the height above the horizon (altitude) in degrees. **(c)** Sketch of the directions of the path (magenta arrow) of Maribo relative to north (N) and of the landing site (M) in Denmark as seen by the camera (C).

Heating of Maribo during its fall in the atmosphere

When the stony meteoroid Maribo entered the atmosphere at supersonic speed it appeared as a fire-ball because the surrounding air was glowing. Nevertheless, only the outermost layer of Maribo was heated. Assume that Maribo is a homogenous sphere with density ρ_{sm}, specific heat capacity c_{sm}, and thermal conductivity k_{sm} (for values see the data sheet). Furthermore, when entering the atmosphere, it had the temperature $T_0 = 200$ K. While falling through the atmosphere its surface temperature was constant $T_{\mathrm{s}} = 1000$ K due to the air friction, thus gradually heating up the interior.

After falling a time t in the atmosphere, an outer shell of Maribo of thickness x will have been heated to a temperature significantly larger than T_0. This thickness can be estimated by dimensional analysis as the simple product of powers of the thermodynamic parameters: $x \approx t^{\alpha} \rho_{\mathrm{sm}}^{\beta}\, c_{\mathrm{sm}}^{\gamma}\, k_{\mathrm{sm}}^{\delta}$.

1.3a	Determine by dimensional (unit) analysis the value of the four powers α, β, γ, and δ.	0.6
1.3b	Calculate the thickness x after a fall time $t = 5$ s, and determine the ratio x/R_{M}.	0.4

The age of a meteorite

The chemical properties of radioactive elements may be different, so during the crystallization of the minerals in a given meteorite, some minerals will have a high content of a specific radioactive element and others a low content. This difference can be used to determine the age of a meteorite by radiometric dating of its radioactive minerals.

As a specific example, we study the isotope ^{87}Rb (element no. 37), which decays into the stable isotope ^{87}Sr (element no. 38) with a half-life of $T_{½} = 4.9 \times 10^{10}$ year, relative to the stable isotope ^{86}Sr. At the time of crystallization the ratio ^{87}Sr/^{86}Sr was identical for all minerals, while the ratio ^{87}Rb/^{86}Sr was different. As time passes on, the amount of ^{87}Rb decreases by decay, while consequently the amount of ^{87}Sr increases. As a result, the ratio ^{87}Sr/^{86}Sr will be different today. In Fig. 1.2(a), the points on the horizontal line refer to the ratio ^{87}Rb/^{86}Sr in different minerals at the time, when they are crystallized.

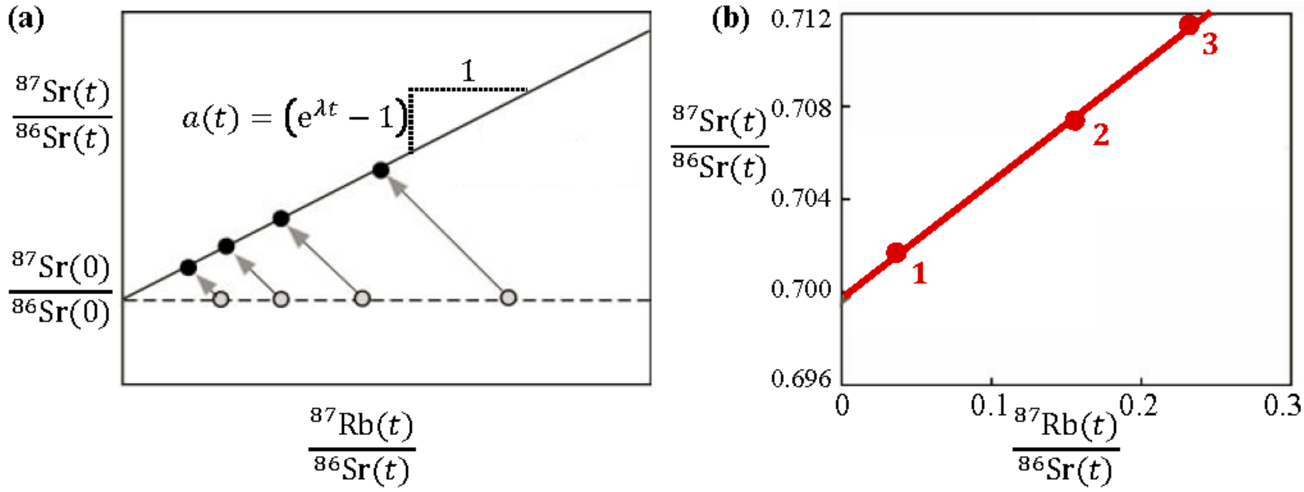

Figure 1.2 **(a)** The ratio ^{87}Sr/^{86}Sr in different minerals at the time $t = 0$ of crystallization (open circles) and at present time (filled circles). **(b)** The isochron-line for three different mineral samples taken from a meteorite at present time.

1.4a	Write down the decay scheme for the transformation of $^{87}_{37}$Rb to $^{87}_{38}$Sr.	0.3
1.4b	Show that the present-time ratio ^{87}Sr/^{86}Sr plotted versus the present-time ratio ^{87}Rb/^{86}Sr in different mineral samples from the same meteorite forms a straight line, the so-called isochron-line, with slope $a(t) = (e^{\lambda t} - 1)$. Here t is the time since the formation of the minerals, while λ is the decay constant inversely proportional to half-life $T_{½}$.	0.7
1.4c	Determine the age τ_{M} of the meteorite using the isochron-line of Fig. 1.2(b).	0.4

Comet Encke, from which Maribo may originate

In its orbit around the Sun, the minimum and maximum distances between comet Encke and the Sun are $a_{\min} = 4.95 \times 10^{10}$ m and $a_{\max} = 6.16 \times 10^{11}$ m, respectively.

1.5	Calculate the orbital period t_{Encke} of comet Encke.	0.6

Consequences of an asteroid impact on Earth

65 million years ago Earth was hit by a huge asteroid with density $\rho_{\text{ast}} = 3.0 \times 10^3\ \text{kg m}^{-3}$, radius $R_{\text{ast}} = 5.0$ km, and final speed of $v_{\text{ast}} = 2.5 \times 10^4$ m/s. This impact resulted in the extermination of most of the life on Earth and the formation of the enormous Chicxulub Crater. Assume that an identical asteroid would hit Earth today in a completely inelastic collision, and use the fact that the moment of inertia of Earth is 0.83 times that for a homogeneous sphere of the same mass and radius. The moment of inertia of a homogeneous sphere with mass M and radius R is $\frac{2}{5}MR^2$. Neglect any changes in the orbit of the Earth.

1.6a	Let the asteroid hit the North Pole. Find the maximum change in angular orientation of the axis of Earth after the impact.	0.7
1.6b	Let the asteroid hit the Equator in a radial impact. Find the change $\Delta\tau_{\text{vrt}}$ in the duration of one revolution of Earth after the impact.	0.7
1.6c	Let the asteroid hit the Equator in a tangential impact in the equatorial plane. Find the change $\Delta\tau_{\tan}$ in the duration of one revolution of Earth after the impact.	0.7

Maximum impact speed

Consider a celestial body, gravitationally bound in the solar system, which impacts the surface of Earth with a speed v_{imp}. Initially the effect of the gravitational field of the Earth on the body can be neglected. Disregard the friction in the atmosphere, the effect of other celestial bodies, and the rotation of the Earth.

1.7	Calculate $v_{\text{imp}}^{\max}$, the largest possible value of v_{imp}.	1.6

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Introduction

In this problem we study an efficient process of steam production that has been demonstrated to work experimentally. An aqueous solution of spherical nanometer-sized silver spheres (nanoparticles) with only about 10^{13} particles per liter is illuminated by a focused light beam. A fraction of the light is absorbed by the nanoparticles, which are heated up and generate steam locally around them without heating up the entire water solution. The steam is released from the system in the form of escaping steam bubbles. Not all details of the process are well understood at present, but the core process is known to be absorption of light through the so-called collective electron oscillations of the metallic nanoparticles. The device is known as a plasmonic steam generator.

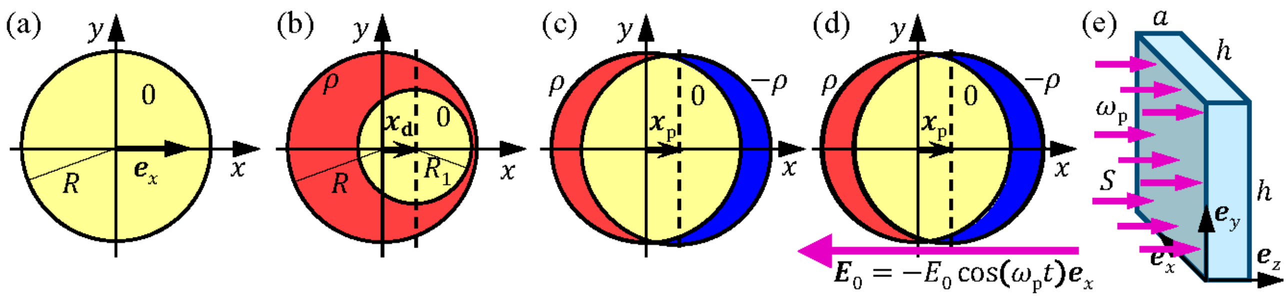

Figure 2.1 **(a)** A spherical charge-neutral nanoparticle of radius R placed at the center of the coordinate system. **(b)** A sphere with a positive homogeneous charge density ρ (red), and containing a smaller spherical charge-neutral region (0, yellow) of radius R_1, with its center displaced by $\boldsymbol{x}_\mathrm{d} = x_\mathrm{d}\,\boldsymbol{e}_x$. **(c)** The sphere with positive charge density ρ of the nanoparticle silver ions is fixed in the center of the coordinate system. The center of the spherical region with negative spherical charge density $-\rho$ (blue) of the electron cloud is displaced by $\boldsymbol{x}_\mathrm{p}$, where $x_\mathrm{p} \ll R$. **(d)** An external homogeneous electric field $\boldsymbol{E}_0 = -E_0\boldsymbol{e}_x$. For time-dependent $\boldsymbol{E}_0$, the electron cloud moves with velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}_\mathrm{p}/\mathrm{d}t$. **(e)** The rectangular vessel ($h \times h \times a$) containing the aqueous solution of nanoparticles illuminated by monochromatic light propagating along the z-axis with angular frequency ω_p and intensity S.

A single spherical silver nanoparticle

Throughout this problem we consider a spherical silver nanoparticle of radius $R = 10{,}0$ nm and with its center fixed at the origin of the coordinate system, see Fig. 2.1(a). All motions, forces and driving fields are parallel to the horizontal x-axis (with unit vector $\boldsymbol{e}_x$). The nanoparticle contains free (conduction) electrons moving within the whole nanoparticle volume without being bound to any silver atom. Each silver atom is a positive ion that has donated one such free electron.

2.1	Find the following quantities: The volume V and mass M of the nanoparticle, the number N and charge density ρ of silver ions in the particle, and for the free electrons their concentration n, their total charge Q, and their total mass m_0.	0.7

The electric field in a charge-neutral region inside a charged sphere

For the rest of the problem assume that the relative dielectric permittivity of all materials is $\varepsilon = 1$. Inside a charged sphere of homogeneous charge density ρ and radius R is created a small spherical charge-neutral region of radius R_1 by adding the opposite charge density $-\rho$, with its center displaced by $\boldsymbol{x}_\mathrm{d} = x_\mathrm{d}\,\boldsymbol{e}_x$ from the center of the R-sphere, see Fig. 2.1(b).

2.2	Show that the electric field inside the charge-neutral region is homogenous of the form $\boldsymbol{E} = A\,(\rho/\varepsilon_0)\,\boldsymbol{x}_\mathrm{d}$, and determine the pre-factor A.	1.2

The restoring force on the displaced electron cloud

In the following, we study the collective motion of the free electrons, and therefore model them as a single negatively charged sphere of homogeneous charge density $-\rho$ with a center position $\boldsymbol{x}_\mathrm{p}$, which can move along the x-axis relative to the center of the positively charged sphere (silver ions) fixed at the origin of the coordinate system, see Fig. 2.1(c). Assume that an external force $\boldsymbol{F}_\mathrm{ext}$ displaces the electron cloud to a new equilibrium position $\boldsymbol{x}_\mathrm{p} = x_\mathrm{p}\,\boldsymbol{e}_x$ with $|x_\mathrm{p}| \ll R$. Except for tiny net charges at opposite ends of the nanoparticle, most of its interior remains charge-neutral.

2.3	Express in terms of $\boldsymbol{x}_\mathrm{p}$ and n the following two quantities: The restoring force $\boldsymbol{F}$ exerted on the electron cloud and the work W_el done on the electron cloud during displacement.	1.0

The spherical silver nanoparticle in an external constant electric field

A nanoparticle is placed in vacuum and influenced by an external force $\boldsymbol{F}_\mathrm{ext}$ due to an applied static homogeneous electric field $\boldsymbol{E}_0 = -E_0\boldsymbol{e}_x$, which displaces the electron cloud the small distance $|x_\mathrm{p}|$, where $|x_\mathrm{p}| \ll R$.

2.4	Find the displacement x_p of the electron cloud in terms of E_0 and n, and determine the amount $-\Delta Q$ of electron charge displaced through the yz-plane at the center of the nanoparticle in terms of n, R and x_p.	0.6

The equivalent capacitance and inductance of the silver nanoparticle

For both a constant and a time-dependent field $\boldsymbol{E}_0$, the nanoparticle can be modeled as an equivalent electric circuit. The equivalent capacitance can be found by relating the work W_el, done on the separation of charges ΔQ, to the energy of a capacitor, carrying charge $\pm\Delta Q$. The charge separation will cause a certain equivalent voltage V_0 across the equivalent capacitor.

2.5a	Express the systems equivalent capacitance C in terms of ε_0 and R, and find its value.	0.7
2.5b	For this capacitance, determine in terms of E_0 and R the equivalent voltage V_0 that should be connected to the equivalent capacitor in order to accumulate the charge ΔQ.	0.4

For a time-dependent field $\boldsymbol{E}_0$, the electron cloud moves with velocity $\boldsymbol{v} = v\,\boldsymbol{e}_x$, Fig. 2.1(d). It has the kinetic energy W_{kin} and forms an electric current I flowing through the fixed yz-plane. The kinetic energy of the electron cloud can be attributed to the energy of an equivalent inductor of inductance L carrying the current I.

2.6a	Express both W_{kin} and I in terms of the velocity v.	0.7
2.6b	Express the equivalent inductance L in terms of particle radius R, the electron charge e and mass m_e, the electron concentration n, and calculate its value.	0.5

The plasmon resonance of the silver nanoparticle

From the above analysis it follows that the motion, arising from displacing the electron cloud from its equilibrium position and then releasing it, can be modeled by an ideal LC-circuit oscillating at resonance. This dynamical mode of the electron cloud is known as the plasmon resonance, which oscillates at the so-called angular plasmon frequency ω_{p}.

2.7a	Find an expression for the angular plasmon frequency ω_{p} of the electron cloud in terms of the electron charge e and mass m_e, the electron density n, and the permittivity ε_0.	0.5
2.7b	Calculate ω_{p} in rad/s and the wavelength λ_{p} in nm of light in vacuum having angular frequency $\omega = \omega_{\text{p}}$.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

In the rest of the problem, the nanoparticle is illuminated by monochromatic light at the angular plasmon frequency ω_{p} with the incident intensity $S = \frac{1}{2}c\varepsilon_0 E_0^2 = 1.00\ \text{MW m}^{-2}$. As the wavelength is large, $\lambda_{\text{p}} \gg R$, the nanoparticle can be considered as being placed in a homogeneous harmonically oscillating field $\boldsymbol{E}_0 = -E_0\cos(\omega_{\text{p}}t)\,\boldsymbol{e}_x$. Driven by $\boldsymbol{E}_0$, the center $\boldsymbol{x}_{\text{p}}(t)$ of the electron cloud oscillates at the same frequency with velocity $\boldsymbol{v} = \text{d}\boldsymbol{x}_{\text{p}}/\text{d}t$ and constant amplitude x_0. This oscillating electron motion leads to absorption of light. The energy captured by the particle is either converted into Joule heating inside the particle or re-emitted by the particle as scattered light.

Joule heating is caused by random inelastic collisions, where any given free electron once in a while hits a silver ion and loses its total kinetic energy, which is converted into vibrations of the silver ions (heat). The average time between the collisions is $\tau \gg 1/\omega_{\text{p}}$, where for silver nanoparticle we use $\tau = 5.24 \times 10^{-15}$ s.

2.8a	Find an expression for the time-averaged Joule heating power P_{heat} in the nanoparticle as well as the time-averaged current squared $\langle I^2 \rangle$, which includes explicitly the time-averaged velocity squared $\langle v^2 \rangle$ of the electron cloud.	1.0
2.8b	Find an expression for the equivalent ohmic resistance R_{heat} in an equivalent resistor-model of the nanoparticle having the Joule heating power P_{heat} due to the electron cloud current I. Calculate the numerical value of R_{heat}.	1.0

The incident light beam loses some time-averaged power P_{scat} by scattering on the oscillating electron cloud (re-emission). P_{scat} depends on the scattering source amplitude x_0, charge Q, angular frequency ω_{p} and properties of the light (the speed of light c and permittivity ε_0 in vacuum). In terms of these four variables, P_{scat} is given by $P_{\text{scat}} = \frac{Q^2 x_0^2 \omega_{\text{p}}{}^4}{12\pi\varepsilon_0 c^3}$.

2.9	By use of P_{scat}, find an expression of the equivalent scattering resistance R_{scat} (in analogy with R_{heat}) in an equivalent resistor-model, and calculate its value.	1.0

The above equivalent circuit elements are combined into an LCR series circuit model of the silver nanoparticle, which is driven by a harmonically oscillating equivalent voltage $V = V_0 \cos(\omega_{\text{p}} t)$ determined by the electric field E_0 of the incident light.

2.10a	Derive expressions for the time-averaged power losses P_{heat} and P_{scat} involving the amplitude E_0 of the electric field in the incident light at the plasmon resonance $\omega = \omega_p$.	1.2
2.10b	Calculate the numerical value of E_0, P_{heat}, and P_{scat}.	0.3

Steam generation by light

An aqueous solution of silver nanoparticles is prepared with a concentration $n_{\text{np}} = 7.3 \times 10^{15}\ \text{m}^{-3}$. It is placed inside a rectangular transparent vessel of size $h \times h \times a = 10 \times 10 \times 1.0\ \text{cm}^3$ and illuminated by light at the plasmon frequency with the same intensity $S = 1.00\ \text{MW m}^{-2}$ at normal incidence as above, see Fig. 2.1(e). The temperature of the water is $T_{\text{wa}} = 20\ ^\circ\text{C}$ and we assume, in fair agreement with observations, that in steady state all Joule heating of the nanoparticle goes to the production of steam of temperature $T_{\text{st}} = 110\ ^\circ\text{C}$, without raising the temperature of the water.

The thermodynamic efficiency η of the plasmonic steam generator is defined by the power ratio $\eta = P_{\text{st}}/P_{\text{tot}}$, where P_{st} is the power going into the production of steam in the entire vessel, while P_{tot} is the total power of the incoming light that enters the vessel.

Most of the time any given nanoparticle is surrounded by steam instead of water, and it can thus be described as being in vacuum.

2.11a	Calculate the total mass per second μ_{st} of steam produced by the plasmonic steam generator during illumination by light at the plasmon frequency and intensity S.	0.6
2.11b	Calculate the numerical value of the thermodynamic efficiency η of the plasmonic steam generator.	0.2

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Average Sun-Earth distance	$a_E = 1.50 \times 10^{11}$ m

Introduction

This problem deals with the physics of the Greenlandic ice sheet, the second largest glacier in the world, Fig. 3.1(a). As an idealization, Greenland is modeled as a rectangular island of width $2L$ and length $5L$ with the ground at sea level and completely covered by incompressible ice (constant density ρ_{ice}), see Fig. 3.1(b). The height profile $H(x)$ of the ice sheet does not depend on the y-coordinate and it increases from zero at the coasts $x = \pm L$ to a maximum height H_{m} along the middle north-south axis (the y-axis), known as the ice divide, see Fig. 3.1(c).

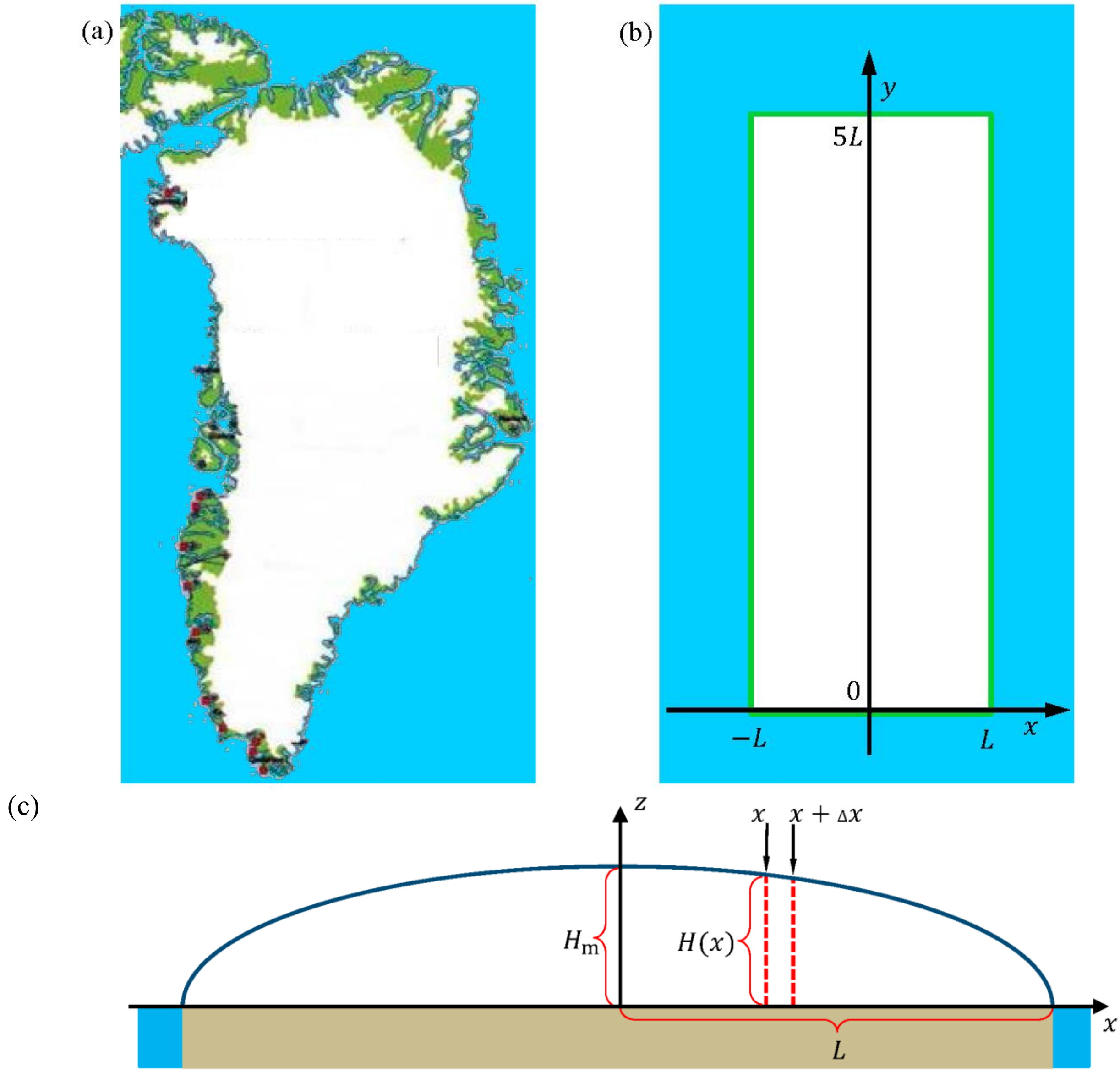

Figure 3.1 (a) A map of Greenland showing the extent of the ice sheet (white), the ice-free, coastal regions (green), and the surrounding ocean (blue). **(b)** The crude model of the Greenlandic ice sheet as covering a rectangular area in the xy-plane with side lengths $2L$ and $5L$. The ice divide, the line of maximum ice sheet height H_{m} runs along the y-axis. **(c)** A vertical cut (xz-plane) through the ice sheet showing the height profile $H(x)$ (blue line). $H(x)$ is independent of the y-coordinate for $0 < y < 5L$, while it drops abruptly to zero at $y = 0$ and $y = 5L$. The z-axis marks the position of the ice divide. For clarity, the vertical dimensions are expanded compared to the horizontal dimensions. The density ρ_{ice} of ice is constant.

Two useful formulas

In this problem you can make use of the integral:

$$\int_0^1 \sqrt{1-x}\,\mathrm{d}x = \frac{2}{3}$$

and the approximation $(1+x)^a \approx 1 + ax$, valid for $|ax| \ll 1$.

The height profile of the ice sheet

On short time scales the glacier is an incompressible hydrostatic system with fixed height profile $H(x)$.

3.1	Write down an expression for the pressure $p(x,z)$ inside the ice sheet as a function of vertical height z above the ground and distance x from the ice divide. Neglect the atmospheric pressure.	0.3

Consider a given vertical slab of the ice sheet in equilibrium, covering a small horizontal base area $\Delta x \Delta y$ between x and $x + \Delta x$, see the red dashed lines in Fig. 3.1(c). The size of Δy does not matter. The net horizontal force component ΔF on the two vertical sides of the slab, arising from the difference in height on the center-side versus the coastal-side of the slab, is balanced by a friction force $\Delta F = S_\mathrm{b}\, \Delta x \Delta y$ from the ground on the base area $\Delta x \Delta y$, where $S_\mathrm{b} = 100$ kPa.

3.2a	For a given value of x, show that in the limit $\Delta x \to 0$, $S_\mathrm{b} = kH\,\mathrm{d}H/\mathrm{d}x$, and determine k	0.9
3.2b	Determine an expression for the height profile $H(x)$ in terms of ρ_ice, g, L, S_b and distance x from the divide. The result will show, that the maximum glacier height H_m scales with the half-width L as $H_\mathrm{m} \propto L^{1/2}$.	0.8
3.2c	Determine the exponent γ with which the total volume V_ice of the ice sheet scales with the area A of the rectangular island, $V_\mathrm{ice} \propto A^\gamma$.	0.5

A dynamical ice sheet

On longer time scale, the ice is a viscous incompressible fluid, which by gravity flows from the center part to the coast. In this model, the ice maintains its height profile $H(x)$ in a steady state, where accumulation of ice due to snow fall in the central region is balanced by melting at the coast. In addition to the ice sheet geometry of Fig. 3.1(b) and (c) make the following model assumptions:

1) Ice flows in the xz-plane away from the ice divide (the y-axis).
2) The accumulation rate c (m/year) in the central region is a constant.
3) Ice can only leave the glacier by melting near the coasts at $x = \pm L$.
4) The horizontal (x-)component $v_x(x) = \mathrm{d}x/\mathrm{d}t$ of the ice-flow velocity is independent of z.
5) The vertical (z-)component $v_z(z) = \mathrm{d}z/\mathrm{d}t$ of the ice-flow velocity is independent of x.

Consider only the central region $|x| \ll L$ close to the middle of the ice sheet, where height variations of the ice sheet are very small and can be neglected altogether, i.e. $H(x) \approx H_\mathrm{m}$.

3.3	Use mass conservation to find an expression for the horizontal ice-flow velocity $v_x(x)$ in terms of c, x, and H_m.	0.6

From the assumption of incompressibility, i.e. the constant density ρ_{ice} of the ice, it follows that mass conservation implies the following restriction on the ice flow velocity components

$$\frac{dv_x}{dx} + \frac{dv_z}{dz} = 0.$$

3.4	Write down an expression for the z dependence of the vertical component $v_z(z)$ of the ice-flow velocity.	0.6

A small ice particle with the initial surface position (x_i, H_{m}) will, as time passes, flow as part of the ice sheet along a flow trajectory $z(x)$ in the vertical xz-plane.

3.5	Derive an expression for such a flow trajectory $z(x)$.	0.9

Age and climate indicators in the dynamical ice sheet

Based on the ice-flow velocity components $v_x(x)$ and $v_z(z)$, one can estimate the age $\tau(z)$ of the ice in a specific depth $H_{\text{m}} - z$ from the surface of the ice sheet.

3.6	Find an expression for the age $\tau(z)$ of the ice as a function of height z above ground, right at the ice divide $x = 0$.	1.0

An ice core drilled in the interior of the Greenland ice sheet will penetrate through layers of snow from the past, and the ice core can be analyzed to reveal past climate changes. One of the best indicators is the so-called $\delta^{18}\text{O}$, defined as

$$\delta^{18}\text{O} = \frac{R_{\text{ice}} - R_{\text{ref}}}{R_{\text{ref}}} 1000\ ‰,$$

where $R = [^{18}\text{O}]/[^{16}\text{O}]$ denotes the relative abundance of the two stable isotopes ^{18}O and ^{16}O of oxygen. The reference R_{ref} is based on the isotopic composition of the oceans around Equator.

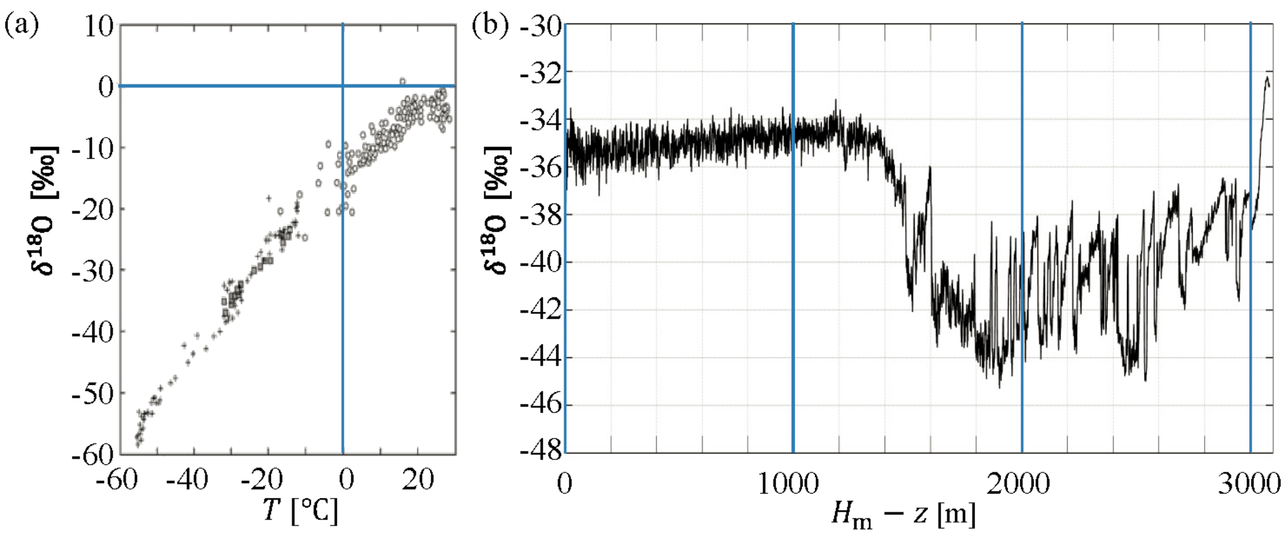

Figure 3.2 **(a)** Observed relationship between $\delta^{18}\text{O}$ in snow versus the mean annual surface temperature T. **(b)** Measurements of $\delta^{18}\text{O}$ versus depth $H_{\text{m}} - z$ from the surface, taken from an ice core drilled from surface to bedrock at a specific place along the Greenlandic ice divide where $H_{\text{m}} = 3060$ m.

Observations from the Greenland ice sheet show that $\delta^{18}\mathrm{O}$ in the snow varies approximately linearly with temperature, Fig. 3.2(a). Assuming that this has always been the case, $\delta^{18}\mathrm{O}$ retrieved from an ice core at depth $H_\mathrm{m} - z$ leads to an estimate of the temperature T near Greenland at the age $\tau(z)$.

Measurements of $\delta^{18}\mathrm{O}$ in a 3060 m long Greenlandic ice core show an abrupt change in $\delta^{18}\mathrm{O}$ at a depth of 1492 m, Fig. 3.2(b), marking the end of the last ice age. The ice age began 120,000 years ago, corresponding to a depth of 3040 m, and the current interglacial age began 11,700 years ago, corresponding to a depth of 1492 m. Assume that these two periods can be described by two different accumulation rates, c_ia (ice age) and c_ig (interglacial age), respectively. You can assume H_m to be constant throughout these 120,000 years.

3.7a	Determine the accumulation rates c_ia and c_ig.	0.8
3.7b	Use the data in Fig. 3.2 to find the temperature change at the transition from the ice age to the interglacial age.	0.2

Sea level rise from melting of the Greenland ice sheet

A complete melting of the Greenlandic ice sheet will cause a sea level rise in the global ocean. As a crude estimate of this sea level rise, one may simply consider a uniform rise throughout a global ocean with constant area $A_\mathrm{O} = 3.61 \times 10^{14}\ \mathrm{m}^2$.

3.8	Calculate the average global sea level rise, which would result from a complete melting of the Greenlandic ice sheet, given its present area of $A_\mathrm{G} = 1.71 \times 10^{12}\mathrm{m}^2$ and $S_\mathrm{b} = 100\ \mathrm{kPa}$.	0.6

The massive Greenland ice sheet exerts a gravitational pull on the surrounding ocean. If the ice sheet melts, this local high tide is lost and the sea level will drop close to Greenland, an effect which partially counteracts the sea level rise calculated above.

To estimate the magnitude of this gravitational pull on the water, the Greenlandic ice sheet is now modeled as a point mass located at the ground level and having the total mass of the Greenlandic ice sheet. Copenhagen lies at a distance of 3500 km along the Earth surface from the center of the point mass. One may consider the Earth, without the point mass, to be spherically symmetric and having a global ocean spread out over the entire surface of the Earth of area $A_\mathrm{E} = 5.10 \times 10^{14}\mathrm{m}^2$. All effects of rotation of the Earth may be neglected.

3.9	Within this model, determine the difference $h_\mathrm{CPH} - h_\mathrm{OPP}$ between sea levels in Copenhagen (h_CPH) and diametrically opposite to Greenland (h_OPP).	1.8