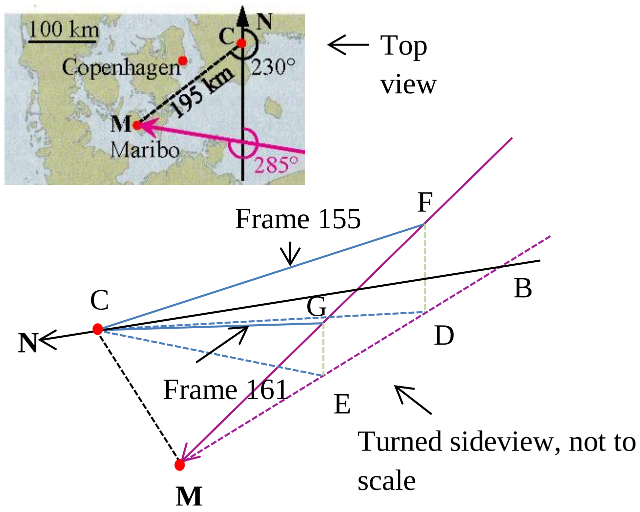
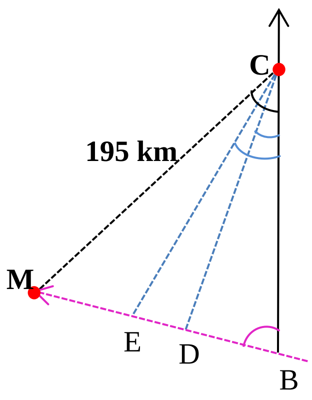
Solutions

1.1	  Top view: Triangle MBC: $\lvert CM\rvert = 195$ km, $\angle BCM = 230° - 180° = 50°$, and $\angle MBC = 75°$, so $\angle CMB = 180° - 75° - 50° = 55°$. Then $\lvert CB\rvert = \frac{\lvert CM\rvert \sin(\angle CMB)}{\sin(\angle MBC)} = 165.4$ km. Triangle DBC: $\lvert BC\rvert = 165.4$ km, $\angle BCD = 215° - 180° = 35°$,and $\angle DBC = 75°$, so $\angle CDB = 180° - 75° - 35° = 70°$. Then $\lvert CD\rvert = \frac{\lvert BC\rvert \sin(\angle DBC)}{\sin(\angle CDB)} = 170.0$ km. Triangle EBC: $\lvert BC\rvert = 165.4$ km, $\angle BCE = 221° - 180° = 41°$, and $\angle EBC = 75°$, so $\angle CEB = 180° - 75° - 41° = 64°$. Then $\lvert CE\rvert = \frac{\lvert BC\rvert \sin(\angle EBC)}{\sin(\angle CEB)} = 177.7$ km. Triangle EDC: $\angle DCE = 41° - 35° = 6°$. Horizontal distance traveled by Maribo: $\lvert DE\rvert = \frac{\lvert DC\rvert \sin(\angle DCE)}{\sin(\angle CED)} = 19.77$ km Side view: Triangle CDF: $\lvert DF\rvert = \lvert DC\rvert \tan(\angle FCD) = 59.20$ km Triangle CEG: $\lvert EG\rvert = \lvert CE\rvert \tan(\angle GCE) = 46.62$ km Thus vertical distance travelled by Maribo: $\lvert DF\rvert - \lvert EG\rvert = 12.57$ km. Total distance travelled by Maribo from frame 155 to 161: $\lvert FG\rvert = \sqrt{\lvert DE\rvert^2 + (\lvert DF\rvert - \lvert EG\rvert)^2} = 23.43$ km. The speed of Maribo is $v = \frac{23.43\text{ km}}{2.28\text{ s} - 1.46\text{ s}} = 28.6$ km/s	1.3

1.2a	Newton's second law: $m_M \frac{dv}{dt} = -k\rho_{atm}\pi R_M^2 v^2$ yields $\frac{1}{v^2} dv = -\frac{k\rho_{atm}\pi R_M^2}{m_M} dt$. By integration $t = \frac{m_M}{k\rho_{atm}\pi R_M^2}\left(\frac{1}{0.9} - 1\right)\frac{1}{v_M} = 0.88$ s. Alternative solution: The average force on the meteoroid when the speed decreases from v_M to $0.9\ v_M$ can be estimated to $F_{av} = -k\rho_{atm}\pi R_M^2\ (0.95\ v_M)^2$. Using that the acceleration is approximately constant, $a_{av} = -k\rho_{atm}\pi R_M^2\ (0.95\ v_M)^2/m_M$, results in $t = \frac{-0.1\ v_M}{a_{av}} = 0.87$ s.	0.7

1.2b	$\frac{E_{\text{kin}}}{E_{\text{melt}}} = \frac{\frac{1}{2}v_{\text{M}}^2}{c_{\text{sm}}(T_{\text{sm}}-T_0)+L_{\text{sm}}} = \frac{4.2\times10^8}{2.1\times10^6} = 2.1\times10^2 \gg 1.$	0.3

1.3a	$[x] = [t]^\alpha[\rho_{\text{sm}}]^\beta[c_{\text{sm}}]^\gamma\,[k_{\text{sm}}]^\delta = [\text{s}]^\alpha[\text{kg m}^{-3}]^\beta[\text{m}^2\,\text{s}^{-2}\text{K}^{-1}]^\gamma\,[\text{kg m s}^{-3}\text{K}^{-1}]^\delta$, so $[\text{m}] = [\text{kg}]^{\beta+\delta}[\text{m}]^{-3\beta+2\gamma+\delta}[\text{s}]^{\alpha-2\gamma-3\delta}[\text{K}]^{-\gamma-\delta}$. Thus $\beta+\delta=0$, $\;-3\beta+2\gamma+\delta=1$, $\;\alpha-2\gamma-3\delta=0$, and $\;-\gamma-\delta=0$. From which $(\alpha,\beta,\gamma,\delta) = \left(+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},+\frac{1}{2}\right)$ and $x(t)\approx\sqrt{\frac{k_{\text{sm}}t}{\rho_{\text{sm}}c_{\text{sm}}}}$.	0.6
1.3b	$x(5\text{ s}) = 1.6\text{ mm}$ $\qquad x/R_{\text{M}} = 1.6\text{ mm}/130\text{ mm} = 0.012.$	0.4

1.4a	Rb-Sr decay scheme: ${}^{87}_{37}\text{Rb} \to {}^{87}_{38}\text{Sr} + {}^{\;0}_{-1}\text{e} + \bar{\nu}_{\text{e}}$	0.3
1.4b	$N_{87\text{Rb}}(t) = N_{87\text{Rb}}(0)\text{e}^{-\lambda t}$ and Rb→Sr: $N_{87\text{Sr}}(t) = N_{87\text{Sr}}(0) + [N_{87\text{Rb}}(0) - N_{87\text{Rb}}(t)]$. Thus $N_{87\text{Sr}}(t) = N_{87\text{Sr}}(0) + (\text{e}^{\lambda t}-1)N_{87\text{Rb}}(t)$, and dividing by $N_{86\text{Sr}}$ we obtain the equation of a straight line: $$\frac{N_{87\text{Sr}}(t)}{N_{86\text{Sr}}} = \frac{N_{87\text{Sr}}(0)}{N_{86\text{Sr}}} + (\text{e}^{\lambda t}-1)\frac{N_{87\text{Rb}}(t)}{N_{86\text{Sr}}}.$$	0.7
1.4c	Slope: $\text{e}^{\lambda t} - 1 = a = \frac{0.712-0.700}{0.25} = 0.050$ and $T_{½} = \frac{\ln(2)}{\lambda} = 4.9\times10^{10}$ year. So $\tau_{\text{M}} = \ln(1+a)\frac{1}{\lambda} = \frac{\ln(1+a)}{\ln(2)}T_{½} = 3.4\times10^9$ year .	0.4

1.5	Kepler's 3rd law on comet Encke and Earth, with the orbital semi-major axis of Encke given by $a = \frac{1}{2}(a_{\min}+a_{\max})$. Thus $t_{\text{Encke}} = \left(\frac{a}{a_{\text{E}}}\right)^{\frac{3}{2}} t_{\text{E}} = 3.30\text{ year} = 1.04\times10^8\text{ s}.$	0.6

1.6a	For Earth around its rotation axis: Angular velocity $\omega_{\text{E}} = \frac{2\pi}{24\text{ h}} = 7.27\times10^{-5}\text{ s}^{-1}$. Moment of inertia $I_{\text{E}} = 0.83\frac{2}{5}m_{\text{E}}R_{\text{E}}^2 = 8.07\times10^{37}\text{ kg m}^2$. Angular momentum $L_{\text{E}} = I_{\text{E}}\omega_{\text{E}} = 5.87\times10^{33}\text{ kg m}^2\text{s}^{-1}$. Asteroid: $m_{\text{ast}} = \frac{4\pi}{3}R_{\text{ast}}^3\rho_{\text{ast}} = 1.57\times10^{15}$ kg and angular momentum $L_{\text{ast}} = m_{\text{ast}}v_{\text{ast}}R_{\text{E}} = 2.51\times10^{26}\text{ kg m}^2\text{s}^{-1}$. L_{ast} is perpendicular to L_{E}, so by conservation angular momentum: $\tan(\Delta\theta) = L_{\text{ast}}/L_{\text{E}} = 4.27\times10^{-8}$.The axis tilt $\Delta\theta = 4.27\times10^{-8}$ rad (so the North Pole moves $R_{\text{E}}\,\Delta\theta = 0.27$ m).	0.7
1.6b	At vertical impact $\Delta L_{\text{E}} = 0$ so $\Delta(I_{\text{E}}\omega_{\text{E}}) = 0$. Thus $\Delta\omega_{\text{E}} = -\omega_{\text{E}}(\Delta I_{\text{E}})/I_{\text{E}}$, and since $\Delta I_{\text{E}}/I_{\text{E}} = m_{\text{ast}}R_{\text{E}}^2/I_{\text{E}} = 7.9\times10^{-10}$ we obtain $\Delta\omega_{\text{E}} = -5.76\times10^{-14}\text{ s}^{-1}$. The change in rotation period is $\Delta T_{\text{E}} = 2\pi\left(\frac{1}{\omega_{\text{E}}+\Delta\omega_{\text{E}}} - \frac{1}{\omega_{\text{E}}}\right) \approx -2\pi\frac{\Delta\omega_{\text{E}}}{\omega_{\text{E}}^2} = 6.84\times10^{-5}$ s.	0.7
1.6c	At tangential impact L_{ast} is parallel to L_{E} so $L_{\text{E}} + L_{\text{ast}} = (I_{\text{E}}+\Delta I_{\text{E}})(\omega_{\text{E}}+\Delta\omega_{\text{E}})$ and thus $\Delta T_{\text{E}} = 2\pi\left(\frac{1}{\omega_{\text{E}}+\Delta\omega_{\text{E}}} - \frac{1}{\omega_{\text{E}}}\right) = 2\pi\left(\frac{I_{\text{E}}+\Delta I_{\text{E}}}{L_{\text{E}}+L_{\text{ast}}} - \frac{1}{\omega_{\text{E}}}\right) = -3.62\times10^{-3}$ s.	0.7

1.7	Maximum impact speed $v_{\mathrm{imp}}^{\max}$ arises from three contributions: (I) The velocity v_{b} of the body at distance a_{E} (Earth orbit radius) from the Sun, $v_{\mathrm{b}} = \sqrt{\frac{2Gm_S}{a_E}} = 42.1\ \mathrm{km/s}$. (II) The orbital velocity of the Earth, $v_{\mathrm{E}} = \frac{2\pi a_E}{1\ \mathrm{year}} = 29.8\ \mathrm{km/s}$. (III) Gravitational attraction from the Earth and kinetic energy seen from the Earth: $\frac{1}{2}(v_{\mathrm{b}} + v_{\mathrm{E}})^2 = -\frac{Gm_E}{R_E} + \frac{1}{2}\left(v_{\mathrm{imp}}^{\max}\right)^2$. In conclusion: $v_{\mathrm{imp}}^{\max} = \sqrt{(v_{\mathrm{b}} + v_{\mathrm{E}})^2 + \frac{2Gm_E}{R_E}} = 72.8\ \mathrm{km/s}$.	1.6

	Total	**9.0**

Solutions

A single spherical silver nanoparticle

2.1	Volume of the nanoparticle: $V=\frac{4}{3}\pi R^3=4.19\times10^{-24}\ \mathrm{m}^3$. Mass of the nanoparticle: $M=V\rho_{\mathrm{Ag}}=4.39\times10^{-20}\ \mathrm{kg}$. Number of ions in the nanoparticle: $N=N_A\frac{M}{M_{\mathrm{Ag}}}=2.45\times10^5$. Charge density $\rho=\frac{eN}{V}=9.38\times10^9\ \mathrm{C\ m^{-3}}$, charge density $\rho=en$. Electrons' concentration $n=\frac{N}{V}=5.85\times10^{28}\ \mathrm{m}^{-3}$. Total charge of free electrons $Q=eN=3.93\times10^{-14}$ C. Total mass of free electrons $m_0=m_eN=2.23\times10^{-25}$ kg.	0.7

The electric field in a charge-neutral region inside a charged sphere

2.2	For a sphere with radius R and constant charge density ρ, for any point inside the sphere designated by radius-vector $\mathbf{r}=r\mathbf{e}_r$ $(r<R)$ Gauss's law yields directly $4\pi r^2\varepsilon_0\boldsymbol{E}_+=\frac{4}{3}\pi r^3\rho\,\boldsymbol{e}_r$, where $\boldsymbol{e}_r$ is the unit radial vector pointing away from the center of the sphere. Thus, $\boldsymbol{E}_+=\frac{\rho}{3\varepsilon_0}\boldsymbol{r}$. Likewise, inside another sphere of radius R_1 and charge density $-\rho$ the field is $\boldsymbol{E}_-=\frac{-\rho}{3\varepsilon_0}\boldsymbol{r}'$, where $\boldsymbol{r}'$ is the radius-vector of the point in the coordinate system with the origin in the center of this sphere. Superposition of the two charge configurations gives the setup we want with $\boldsymbol{r}'=\boldsymbol{r}-\boldsymbol{x}_d$. So inside the charge-free region $\lvert\boldsymbol{r}-\boldsymbol{x}_{\mathrm{p}}\rvert<R_1$ the field is $\boldsymbol{E}=\boldsymbol{E}_++\boldsymbol{E}_-=\frac{\rho}{3\varepsilon_0}\mathbf{r}+\frac{-\rho}{3\varepsilon_0}(\boldsymbol{r}-\boldsymbol{x}_d)$ or $\boldsymbol{E}=\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_d$ with pre-factor $A=\frac{1}{3}$	1.2

The restoring force on the displaced electron cloud

2.3	With $\boldsymbol{x}_{\mathrm{p}}=x_{\mathrm{p}}\,\boldsymbol{e}_x$ and $x_{\mathrm{p}}\ll R$ we have from above that approximately the field induced inside the particle is $\boldsymbol{E}_{\mathrm{ind}}=\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}}$. The number of electrons on the particle's border that produced $\boldsymbol{E}_{\mathrm{ind}}$ is negligibly smaller than the number of electrons inside the particle, so $\boldsymbol{F}\cong Q\boldsymbol{E}_{\mathrm{ind}}=(-eN)\frac{\rho}{3\varepsilon_0}\boldsymbol{x}_{\mathrm{p}}=-\frac{4\pi}{9\varepsilon_0}R^3e^2n^2\,x_p\boldsymbol{e}_x$ (note the antiparallel attractive force is proportional to the displacement that it is similar to Hooke's law). The work done on the electron cloud to shift it is $W_{\mathrm{el}}=-\int_0^{x_{\mathrm{p}}}F(x')\,\mathrm{d}x'=\frac{1}{2}\left(\frac{4\pi}{9\varepsilon_0}R^3e^2n^2\right)x_{\mathrm{p}}^2$	1.0

The spherical silver nanoparticle in an external constant electric field

2.4	Inside the metallic particle in the steady state the electric field must be equal to 0. The induced field (from 2.2 or 2.3) compensates the external field: $\boldsymbol{E}_0+\boldsymbol{E}_{\mathrm{ind}}=0$, so	0.6

	$x_\mathrm{p} = \frac{3\varepsilon_0}{\rho}E_0 = \frac{3\varepsilon_0}{en}E_0$. Charge displaced through the yz-plane is the total charge of electrons in the cylinder of radius R and height x_p: $-\Delta Q = -\rho\,\pi R^2 x_\mathrm{p} = -\pi R^2\, ne\, x_\mathrm{p}$.	

The equivalent capacitance and inductance of the silver nanoparticle

2.5a	The electric energy W_el of a capacitor with capacitance C holding charges $\pm\Delta Q$ is $W_\mathrm{el} = \frac{\Delta Q^2}{2C}$. The energy of such capacitor is equal to the work (see 2.3) done to separate the charges (see 2.4), thus $C = \frac{\Delta Q^2}{2W_{el}} = \frac{9}{4}\varepsilon_0\pi R = 6.26\times 10^{-19}$ F.	0.7
2.5b	Equivalent scheme for a capacitor reads: $\Delta Q = CV_0$. Combining charge from (2.4) and capacitance from (2.5a) gives $V_0 = \frac{\Delta Q}{C} = \frac{4}{3}R\,E_0$.	0.4

2.6a	The kinetic energy of the electron cloud is defined as the kinetic energy of one electron multiplied by the number of electrons in the cloud $W_\mathrm{kin} = \frac{1}{2}m_e v^2 N = \frac{1}{2}m_e v^2\left(\frac{4}{3}\pi R^3\, n\right)$. The current I is the charge of electrons in the cylinder of area πR^2 and height $v\Delta t$ divided by time Δt (or simply the time derivative of charge $-\Delta Q$), thus $I = -e\,nv\,\pi R^2$.	0.7
2.6b	The energy carried by current I in the equivalent circuit with inductance L is $W = \frac{1}{2}L\,I^2$ is, in fact, the kinetic energy of electrons W_kin. Taking the energy and current from (2.6a) gives $L = \frac{4\,m_e}{3\pi R n e^2} = 2.57\times 10^{-14}$ H.	0.5

The plasmon resonance of the silver nanoparticle

2.7a	From the LC-circuit analogy we can directly derive $\omega_p = (LC)^{-1/2} = \sqrt{ne^2/3\varepsilon_0 m_e}$. Alternatively it is possible to use the harmonic law of motion in (2.3) and get the same result for the frequency.	0.5
2.7b	$\omega_\mathrm{p} = 7.88\times 10^{15}$ rad/s, for light with angular frequency $\omega = \omega_\mathrm{p}$ the wavelength is $\lambda_\mathrm{p} = 2\pi c/\omega_\mathrm{p} = 239$ nm.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

2.8a	The velocity of an electron $v = \frac{dx}{dt} = -\omega x_0 \sin\omega t = v_0 \sin\omega t$. The time-averaged kinetic energy on the electron $\langle W_k\rangle = \langle\frac{m_e v^2}{2}\rangle = \frac{m_e}{2}\langle v^2\rangle$. During time τ each electron hits an ion one time. So the energy lost in the whole nanoparticle during time τ is $W_{heat} = N\langle\frac{m_e v^2}{2}\rangle = \frac{4}{3}\pi R^3 n\langle\frac{m_e v^2}{2}\rangle$. Time-averaged Joule heating power $P_\mathrm{heat} = \frac{1}{\tau}W_\mathrm{kin} = \frac{1}{2\tau}m_e\langle v^2\rangle\left(\frac{4}{3}\pi R^3\, n\right)$. The expression for current is taken from (2.6a), squared and averaged	1.0

	$\langle I^2\rangle = (en\,\pi R^2)^2\,\langle v^2\rangle = \left(\frac{3Q}{4R}\right)^2\langle v^2\rangle$.	
2.8b	The average time between the collisions is $\tau \gg 1/\omega_\mathrm{p}$, so each electron oscillates many times before it collides with an ion. The oscillating current $I = I_0\sin\omega t = \pi R^2 nev_0\sin\omega t$ produces the heat in the resistance R_{heat} equal to $P_\mathrm{heat} = R_{heat}\langle I^2\rangle$, that together with results from (2.8a) leads to $R_\mathrm{heat} = \frac{W_\mathrm{kin}}{\tau\langle I^2\rangle} = \frac{2m_e}{3\pi ne^2R\tau} = 2.46\ \Omega$.	1.0

2.9	For equivalent scattering resistance $R_\mathrm{scat} = \frac{P_\mathrm{scat}}{\langle I^2\rangle}$ and for harmonic oscillations we can average the velocity squared over one period of oscillations, so $\langle v^2\rangle = \frac{1}{2}\omega_\mathrm{p}^2 x_0^2$. Together it yields $R_\mathrm{scat} = \frac{Q^2x_0^2\omega_\mathrm{p}^4}{12\pi\varepsilon_0 c^3}\frac{16R^2}{9Q^2\langle v^2\rangle} = \frac{8\omega_0^2R^2}{27\pi\varepsilon_0c^3} = 2.45\ \Omega$.	1.0

2.10a	Ohm's law for a LCR serious circuit is $I_0 = \frac{V_0}{\sqrt{(R_{heat}+R_{scat})^2+\left(\omega L-\frac{1}{\omega C}\right)^2}}$. At the resonance frequency time-averaged voltage squared is $\langle V^2\rangle = Z_R^2\langle I^2\rangle = (R_\mathrm{heat}+R_\mathrm{scat})^2\langle I^2\rangle$. And from (2.5b) $\langle V^2\rangle = \frac{1}{2}V_0^2 = \frac{8}{9}R^2E_0^2$, so Ohm's law results in $\langle I^2\rangle = \frac{8R^2E_0^2}{9(R_\mathrm{heat}+R_\mathrm{scat})^2}$. The time-averaged power losses are $P_\mathrm{heat} = R_\mathrm{heat}\langle I^2\rangle = \frac{8R_\mathrm{heat}R^2}{9(R_\mathrm{heat}+R_\mathrm{scat})^2}E_0^2$ and $P_\mathrm{scat} = \frac{8R_\mathrm{scat}R^2}{9(R_\mathrm{heat}+R_\mathrm{scat})^2}E_0^2 = \frac{R_\mathrm{scat}}{R_\mathrm{heat}}\langle P_\mathrm{heat}\rangle$.	1.2
2.10b	Starting with the electric field amplitude $E_0 = \sqrt{2S/(\varepsilon_0 c)} = 27.4$ kV/m, we calculate $P_\mathrm{heat} = 6.82$ nW and $P_\mathrm{scat} = 6.81$ nW.	0.3

Steam generation by light

2.11a	Total number of nanoparticles in the vessel: $N_\mathrm{np} = h^2a\,n_\mathrm{np} = 7.3\times10^{11}$. Then the total time-averaged Joule heating power: $P_\mathrm{st} = N_\mathrm{np}P_\mathrm{heat} = 4.98$ kW. This power goes into the steam generation: $P_\mathrm{st} = \mu_\mathrm{st}L_\mathrm{tot}$, with $L_\mathrm{tot} = c_\mathrm{wa}(T_{100}-T_\mathrm{wa}) + L_\mathrm{wa} + c_\mathrm{st}(T_\mathrm{st}-T_{100}) = 2.62\times10^6\ \mathrm{J\,kg^{-1}}$. Thus the mass of steam produced in one second is: $\mu_\mathrm{st} = \frac{P_\mathrm{st}}{L_\mathrm{tot}} = 1.90\times10^{-3}\ \mathrm{kg\,s^{-1}}$.	0.6
2.11b	The power of light incident on the vessel $P_\mathrm{tot} = h^2S = 0.01\mathrm{m}^2\times 1\ \mathrm{MW\,m^{-2}} = 10.0$ kW, and the power directed for steam production by nanoparticles is given in 2.11a. Efficiency of the process is $\eta = \frac{P_\mathrm{st}}{P_\mathrm{tot}} = \frac{4.98\ \mathrm{kW}}{10.0\ \mathrm{kW}} = 0.498$.	0.2

	Total	**12.0**

Solutions

3.1	The pressure is given by the hydrostatic pressure $p(x,z) = \rho_{\text{ice}} g(H(x) - z)$, which is zero at the surface.	0.3
3.2a	The outward force on a vertical slice at a distance x from the middle and of a given width Δy is obtained by integrating up the pressure times the area: $$F(x) = \Delta y \int_0^{H(x)} \rho_{\text{ice}}\, g\, (H(x) - z)\, \mathrm{d}z = \frac{1}{2}\, \Delta y\, \rho_{\text{ice}}\, g\, H(x)^2$$ which implies that $\Delta F = F(x) - F(x + \Delta x) = -\frac{\mathrm{d}F}{\mathrm{d}x}\Delta x = -\Delta y\, \rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x} \Delta x$. This finally shows that $$S_{\mathrm{b}} = \frac{\Delta F}{\Delta x \Delta y} = -\,\rho_{\text{ice}}\, g\, H(x) \frac{\mathrm{d}H}{\mathrm{d}x}$$ Notice the sign, which must be like this, since S_b was defined as positive and $H(x)$ is a decreasing function of x.	0.9
3.2b	To find the height profile, we solve the differential equation for $H(x)$: $$-\frac{S_{\mathrm{b}}}{\rho_{\text{ice}}\, g} = H(x)\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x} H(x)^2$$ with the boundary condition that $H(L) = 0$. This gives the solution: $$H(x) = \sqrt{\frac{2 S_b L}{\rho_{\text{ice}}\, g}} \sqrt{1 - x/L}$$ Which gives the maximum height $H_{\mathrm{m}} = \sqrt{\frac{2S_b L}{\rho_{\text{ice}}\, g}}$. Alternatively, dimensional analysis could be used in the following manner. First notice that $\mathcal{L} = [H_{\mathrm{m}}] = [\rho_{\text{ice}}^{\alpha}\, g^{\beta} \tau_{\mathrm{b}}^{\gamma} L^{\delta}]$. Using that $[\rho_{\rho_{\text{ice}}}] = \mathcal{M}\mathcal{L}^{-3}$, $[g] = \mathcal{L}\,\mathcal{T}^{-2}$, $[\tau_b] = \mathcal{M}\mathcal{L}^{-1}\,\mathcal{T}^{-2}$, demands that $\mathcal{L} = [H_{\mathrm{m}}] = [\rho_i{}^{\alpha} g^{\beta} \tau_b{}^{\gamma} L^{\delta}] = \mathcal{M}^{\alpha+\gamma} \mathcal{L}^{-3\alpha+\beta-\gamma+\delta}\, \mathcal{T}^{-2\beta-2\gamma}$, which again implies $\alpha + \gamma = 0,\ -3\alpha + \beta - \gamma + \delta = 1,\ 2\beta + 2\gamma = 0$. These three equations are solved to give $\alpha = \beta = -\gamma = \delta - 1$, which shows that $$H_{\mathrm{m}} \propto \left(\frac{S_{\mathrm{b}}}{\rho_{\rho_{\text{ice}}} g}\right)^{\gamma} L^{1-\gamma}$$ Since we were informed that $H_{\mathrm{m}} \propto \sqrt{L}$, it follows that $\gamma = 1/2$. With the boundary condition $H(L) = 0$, the solution then take the form $$H(x) \propto \left(\frac{S_{\mathrm{b}}}{\rho_{\text{ice}}\, g}\right)^{1/2} \sqrt{L - x}$$ The proportionality constant of $\sqrt{2}$ cannot be determined in this approach.	0.8

3.2c	For the rectangular Greenland model, the area is equal to $A = 10L^2$ and the volume is found by integrating up the height profile found in problem 3.2b: $V_{\mathrm{G,ice}} = (5L)2\int_0^L H(x)\,\mathrm{d}x = 10L\int_0^L \left(\frac{\tau_{\mathrm{b}}L}{\rho_{\mathrm{ice}}\,g}\right)^{1/2}\sqrt{1-x/L}\,\mathrm{d}x = 10H_{\mathrm{m}}L^2\int_0^1\sqrt{1-\tilde{x}}\,\mathrm{d}\tilde{x}$ $= 10H_{\mathrm{m}}L^2\left[-\frac{2}{3}(1-\tilde{x})^{3/2}\right]_0^1 = \frac{20}{3}H_{\mathrm{m}}L^2 \propto L^{5/2}$, where the last line follows from the fact that $H_{\mathrm{m}} \propto \sqrt{L}$. Note that the integral need not be carried out to find the scaling with L. This implies that $V_{\mathrm{G,ice}} \propto {A_G}^{5/4}$ and the wanted exponent is $\gamma = 5/4$.	0.5

3.3	According to the assumption of constant accumulation c the total mass accumulation rate from an area of width Δy between the ice divide at $x = 0$ and some point at $x > 0$ must equal the total mass flux through the corresponding vertical cross section at x. That is: $\rho c x \Delta y = \rho \Delta y H_{\mathrm{m}} v_x(x)$, from which the velocity is isolated: $$v_x(x) = \frac{cx}{H_{\mathrm{m}}}$$	0.6

3.4	From the given relation of incompressibility it follows that $$\frac{\mathrm{d}v_z}{\mathrm{d}z} = -\frac{\mathrm{d}v_x}{\mathrm{d}x} = -\frac{c}{H_{\mathrm{m}}}$$ Solving this differential equation with the initial condition $v_z(0) = 0$, shows that: $$v_z(z) = -\frac{cz}{H_{\mathrm{m}}}$$	0.6

3.5	Solving the two differential equations $$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{cz}{H_{\mathrm{m}}} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{cx}{H_{\mathrm{m}}}$$ with the initial conditions that $z(0) = H_{\mathrm{m}}$, and $x(0) = x_i$ gives $$z(t) = H_{\mathrm{m}}\,\mathrm{e}^{-ct/H_{\mathrm{m}}} \quad \text{and} \quad x(t) = x_i\,\mathrm{e}^{ct/H_{\mathrm{m}}}$$ This shows that $z = H_{\mathrm{m}}\,x_i\,/x$, meaning that flow lines are hyperbolas in the xz-plane. Rather than solving the differential equations, one can also use them to show that $$\frac{\mathrm{d}}{\mathrm{d}t}(xz) = \frac{\mathrm{d}x}{\mathrm{d}t}z + x\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{cx}{H_{\mathrm{m}}}z - x\frac{cz}{H_{\mathrm{m}}} = 0$$ which again implies that $xz =$ const. Fixing the constant by the initial conditions, again leads to the result that $z = H_{\mathrm{m}}x_i/x$.	0.9

3.6	At the ice divide, $x = 0$, the flow will be completely vertical, and the t-dependence of z found in 3.5 can be inverted to find $\tau(z)$. One finds that $\tau(z) = \frac{H_{\mathrm{m}}}{c}\ln\left(\frac{H_{\mathrm{m}}}{z}\right)$.	1.0

3.7a	The present interglacial period extends to a depth of 1492 m, corresponding to 11,700 year. Using the formula for $\tau(z)$from problem 3.6, one finds the following accumulation rate for the interglacial: $$c_{\mathrm{ig}} = \frac{H_\mathrm{m}}{11{,}700\ \mathrm{years}} \ln\left(\frac{H_\mathrm{m}}{H_\mathrm{m} - 1492\ \mathrm{m}}\right) = 0.1749\ \mathrm{m/year}.$$ The beginning of the ice age 120,000 years ago is identified as the drop in $\delta^{18}\mathrm{O}$ in figure 3.2b at a depth of 3040 m. Using the vertical flow velocity found in problem 3.4, on has $\frac{\mathrm{d}z}{z} = -\frac{c}{H_\mathrm{m}}\mathrm{d}t$, which can be integrated down to a depth of 3040 m, using a stepwise constant accumulation rate: $$H_\mathrm{m} \ln\left(\frac{H_\mathrm{m}}{H_\mathrm{m} - 3040\ \mathrm{m}}\right) = -H_\mathrm{m}\int_{H_\mathrm{m}}^{H_\mathrm{m}-3040\ \mathrm{m}} \frac{1}{z}\mathrm{d}z$$ $$= \int_{11{,}700\ \mathrm{year}}^{120{,}000\ \mathrm{year}} c_{\mathrm{ia}}\ \mathrm{d}t + \int_0^{11{,}700\ \mathrm{year}} c_{\mathrm{ig}}\ \mathrm{d}t$$ $$= c_{\mathrm{ia}}(120{,}000\ \text{year-}11{,}700\ \text{year}) + c_{\mathrm{ig}} 11{,}700\ \mathrm{year}$$ Isolating form this equation leads to $c_{\mathrm{ia}} = 0.1232$, i.e. far less precipitation than now.	0.8
3.7b	Reading off from figure 3.2b: $\delta^{18}\mathrm{O}$ changes from $-43{,}5$ ‰ to $-34{,}5$ ‰. Reading off from figure 3.2a, T then changes from -40 °C to -28 °C. This gives $\Delta T \approx 12$ °C.	0.2

3.8	From the area A_G one finds that $L = \sqrt{A_\mathrm{G}/10} = 4.14 \times 10^5$ m. Inserting numbers in the volume formula found in 3.2c, one finds that: $$V_{\mathrm{G,ice}} = \frac{20}{3} L^{5/2} \sqrt{\frac{2S_\mathrm{b}}{\rho_{\mathrm{ice}} g}} = 3.45 \times 10^{15}\ \mathrm{m}^3$$ This ice volume must be converted to liquid water volume, by equating the total masses, i.e. $V_{\mathrm{G,wa}} = V_{\mathrm{G,ice}} \frac{\rho_{\mathrm{ice}}}{\rho_{\mathrm{wa}}} = 3.17 \times 10^{15}\ \mathrm{m}^3$, which is finally converted to a sea level rise, as $h_{\mathrm{G,rise}} = \frac{V_{\mathrm{G,wa}}}{A_\mathrm{o}} = 8.79$ m.	0.6

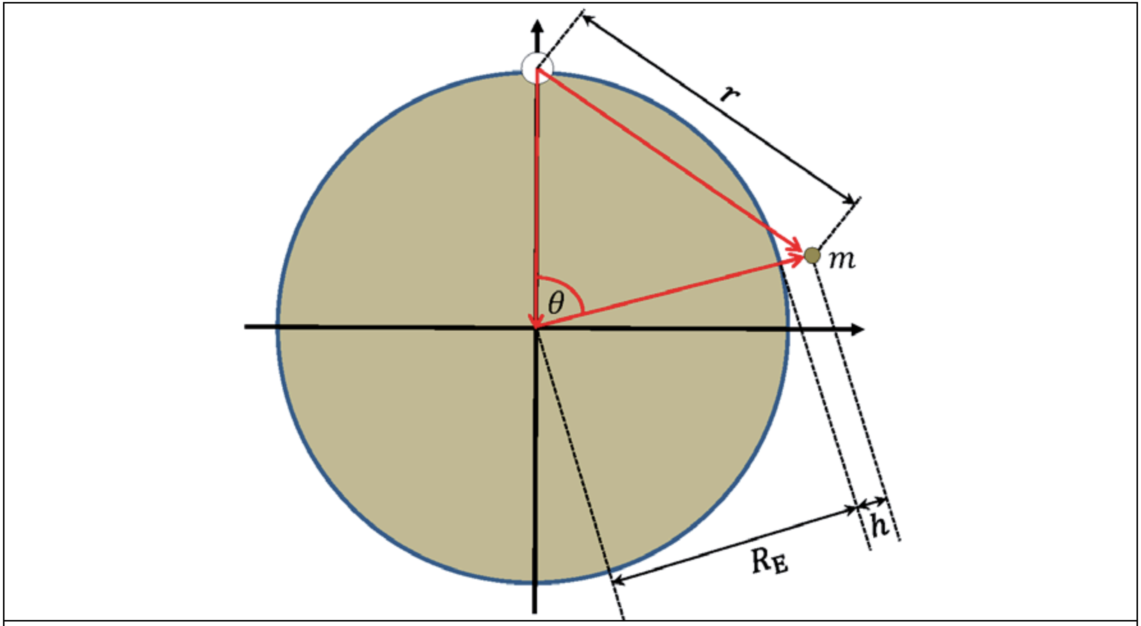


Figure 3.S1 Geometry of the ice ball (white circle) with a test mass m (small gray circle).

3.9	The total mass of the ice is $$M_{\text{ice}} = V_{\text{G,ice}}\,\rho_{\text{ice}} = 3.17\times10^{18}\ \text{kg} = 5.31\times10^{-7} m_{\text{E}}$$ The total gravitational potential felt by a test mass m at a certain height h above the surface of the Earth, and at a polar angle θ (cf. figure 3.S1), with respect to a rotated polar axis going straight through the ice sphere is found by adding that from the Earth with that from the ice: $$U_{\text{tot}} = -\frac{Gm_{\text{E}}m}{R_{\text{E}}+h} - \frac{GM_{\text{ice}}m}{r} = -mgR_E\left(\frac{1}{1+h/R_E} + \frac{M_{ice}/m_E}{r/R_E}\right)$$ where $g = Gm_E/R_E^2$. Since $h/R_{\text{E}} \ll 1$ one may use the approximation given in the problem, $(1+\text{x})^{-1} \approx 1-x,\ \ \lvert x\rvert \ll 1$, to approximate this by $$U_{\text{tot}} \approx -mgR_E\left(1 - \frac{h}{R_E} + \frac{M_{ice}/m_E}{r/R_E}\right).$$ Isolating h now shows that $h = h_0 + \frac{M_{ice}/m_E}{r/R_E}R_E$, where $h_0 = R_E + U_{\text{tot}}/(mg)$. Using again that $h/R_{\text{E}} \ll 1$, trigonometry shows that $r \approx 2R_{\text{E}}\lvert\sin(\theta/2)\rvert$, and one has: $$h(\theta) - h_0 \approx \frac{M_{\text{ice}}/m_{\text{E}}}{2\lvert\sin(\theta/2)\rvert}R_E \approx \frac{1.69\ \text{m}}{\lvert\sin(\theta/2)\rvert}.$$ To find the magnitude of the effect in Copenhagen, the distance of 3500 km along the surface is used to find the angle $\theta_{\text{CPH}} = (3.5\times10^6\ \text{m})/R_E \approx 0.549$, corresponding to $h_{\text{CPH}} - h_0 \approx 6.25$ m. Directly opposite to Greenland corresponds to $\theta = \pi$, which gives $h_{\text{OPP}} - h_0 \approx 1.69$ m. The difference is then $h_{\text{CPH}} - h_{\text{OPP}} \approx 4.56$ m, where h_0 has dropped out.	1.6

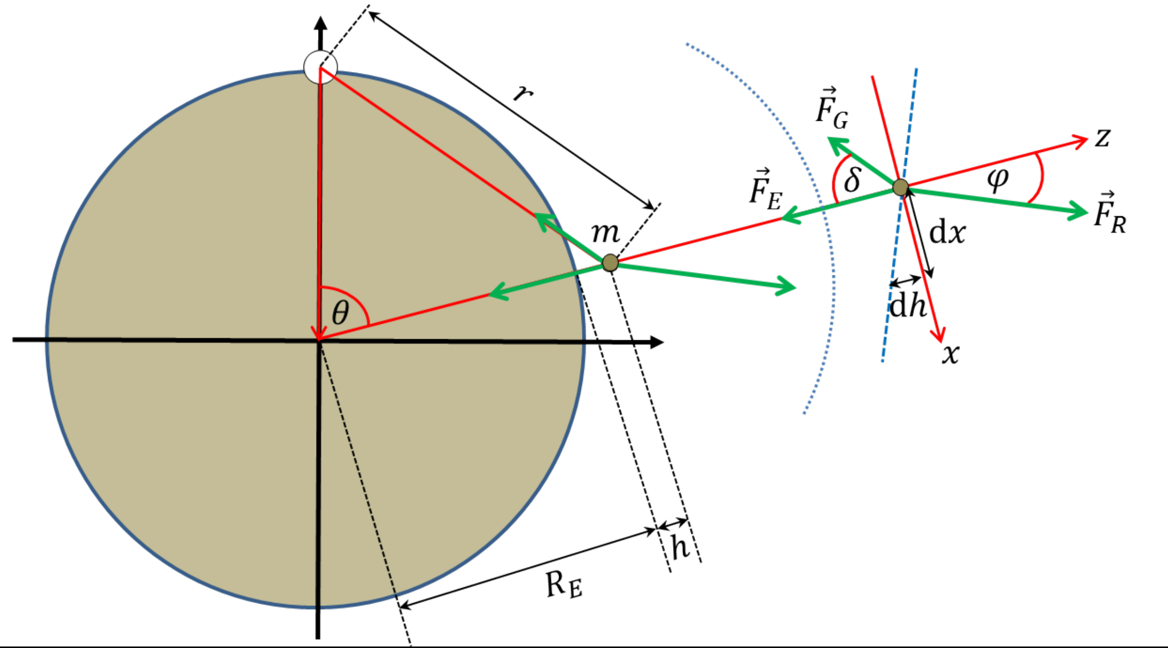


Figure 3.S2 Same figure as above, but with the relevant forces depicted and showed again outside figure for clarity. The blue dotted line indicates the Earth surface. The blue dashed line indicates the local sea level, growing towards Greenland and decreasing towards the south pole.

Approach with forces:

This problem can also be solved using forces. The basic equations for mechanical equilibrium of the test particle is then a simple matter of balancing the two gravitational forces, $\vec{F}_E$ and $\vec{F}_G$, with the reaction force from the Earth, $\vec{F}_R$. Given the angles indicated in Figure 3.S2, the force balance along locally vertical and horizontal directions, respectively, read

$$F_E + F_G\cos(\delta) = F_R\cos(\varphi)$$

and

$$F_G\sin(\delta) = F_R\sin(\varphi)$$

which can be divided to obtain (using that $\delta = \pi/2 - \theta/2$):

$$\begin{aligned}\tan(\varphi) &= \frac{F_G\sin(\delta)}{F_E + F_G\cos(\delta)}\\ &= \frac{F_G}{F_E}\cos(\theta/2)\frac{1}{1+(F_G/F_E)\sin(\theta/2)}\\ &\approx \frac{F_G}{F_E}\cos(\theta/2)\\ &= \frac{M_{ice}/m_E}{(r/R_E)^2}\cos(\theta/2)\\ &= \frac{M_{ice}/m_E}{4\sin^2(\theta/2)}\cos(\theta/2)\end{aligned}$$

where we have plugged in the gravitational forces and the relevant distances. We have also

	approximated the fraction, using that $M_{ice}/m_E = 5.31\times 10^{-7} \ll 1$, which is only valid not too close to Greenland, i.e. for a certain size of θ. Since the local sea surface will be perpendicular to the reaction force, it is seen from figure 3.S2 that $$\tan(\varphi) = \frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{R_E}\frac{\mathrm{d}h}{\mathrm{d}\theta}$$ whereby $$\frac{\mathrm{d}h}{\mathrm{d}\theta} = R_E \frac{M_{ice}/m_E}{4\sin^2(\theta/2)}\cos(\theta/2)$$ The difference in sea levels in Copenhagen and opposite to Greenland can now be obtained by integrating this expression. That is $$\begin{aligned} h_{\mathrm{CPH}} - h_{\mathrm{OPP}} &= R_E \frac{M_{ice}}{m_E}\int_{\pi}^{\theta_{CPH}} \frac{\cos(\theta/2)}{4\sin^2(\theta/2)}\,\mathrm{d}\theta \\ &= R_E \frac{M_{ice}}{2\,m_E}\int_{1}^{\sin(\theta_{CPH}/2)} \mathrm{q}^{-2}\,\mathrm{d}q \\ &= R_E \frac{M_{ice}}{2\,m_E}\left(\frac{1}{\sin(\theta_{CPH}/2)} - 1\right) \end{aligned}$$ where we have made the substitution $q = \sin(\theta/2)$. Plugging in the numbers found above, we obtain again $h_{\mathrm{CPH}} - h_{\mathrm{OPP}} \approx 4.56$. Note that this solution strategy necessarily involves consideration of tangential force components alongside with the radial components.	

	Total	**9.0**