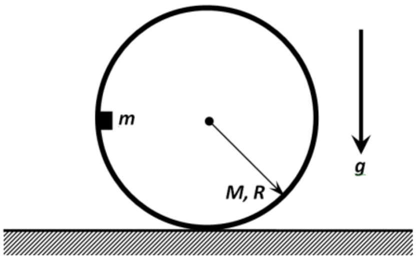
Problem 1 (9 points)

This problem consists of three independent parts.

Part A (3 points)



A small puck of mass m is carefully placed onto the inner surface of the thin hollow thin cylinder of mass M and of radius R. Initially, the cylinder rests on the horizontal plane and the puck is located at the height R above the plane as shown in the figure on the left. Find the interaction force F between the puck and the cylinder at the moment when the puck passes the lowest point of its trajectory. Assume that the friction between the puck and the inner surface of the cylinder is absent, and the cylinder moves on the plane without slipping. The free fall acceleration is g.

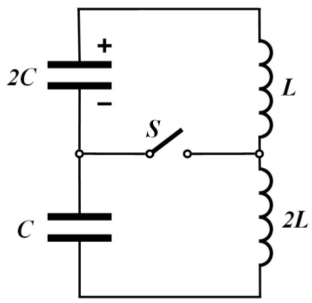
Part B (3 points)

A bubble of radius $r = 5.00$ cm, containing a diatomic ideal gas, has the soap film of thickness $h = 10.0$ μm and is placed in vacuum. The soap film has the surface tension $\sigma = 4.00 \cdot 10^{-2}\ \frac{\mathrm{N}}{\mathrm{m}}$ and the density $\rho = 1.10\ \frac{\mathrm{g}}{\mathrm{cm}^3}$. 1) Find formula for the molar heat capacity of the gas in the bubble for such a process when the gas is heated so slowly that the bubble remains in a mechanical equilibrium and evaluate it; 2) Find formula for the frequency ω of the small radial oscillations of the bubble and evaluate it under the assumption that the heat capacity of the soap film is much greater than the heat capacity of the gas in the bubble. Assume that the thermal equilibrium inside the bubble is reached much faster than the period of oscillations.

Hint: Laplace showed that there is pressure difference between inside and outside of a curved surface, caused by surface tension of the interface between liquid and gas, so that $\Delta p = \frac{2\sigma}{r}$.

Part C (3 points)

Initially, a switch S is unshorted in the circuit shown in the figure on the right, a capacitor of capacitance $2C$ carries the electric charge q_0, a capacitor of capacitance C is uncharged, and there are no electric currents in both coils of inductance L and $2L$, respectively. The capacitor starts to discharge and at the moment when the current in the coils reaches its maximum value, the switch S is instantly shorted. Find the maximum current $I_{\max}$ through the switch S thereafter.



Problem 2. Van der Waals equation of state (11 points)

In a well-known model of an ideal gas, whose equation of state obeys the Clapeyron-Mendeleev law, the following important physical effects are neglected. First, molecules of a real gas have a finite size and, secondly, they interact with one another. In all parts of this problem *one mole of water* is considered.

Part A. Non-ideal gas equation of state (2 points)

Taking into account the finite size of the molecules, the gaseous equation of state takes the form

$$P(V-b)=RT, \quad (1)$$

where P, V, T stands for the gas pressure, its volume per mole and temperature, respectively, R denotes the universal gas constant, and b is a specific constant extracting some volume.

A1	Estimate b and express it in terms of the diameter of the molecules d. **(0.3 points)**

With account of intermolecular attraction forces, van der Waals proposed the following equation of state that neatly describes both the gaseous and liquid states of matter

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT. \quad (2)$$

where a is another specific constant.

At temperatures T below a certain critical value T_c the isotherm of equation (2) is well represented by a non-monotonic curve 1 shown in Figure 1 which is then called van der Waals isotherm. In the same figure curve 2 shows the isotherm of an ideal gas at the same temperature. A real isotherm differs from the van der Waals isotherm by a straight segment ***AB*** drawn at some constant pressure P_{LG}. This straight segment is located between the volumes V_L and V_G, and corresponds to the equilibrium of the liquid phase (indicated by L) and the gaseous phase (referred to by G). From the second law of thermodynamics J. Maxwell showed that the pressure P_{LG} must be chosen such that the areas *I* and *II* shown in Figure 1 must be equal.

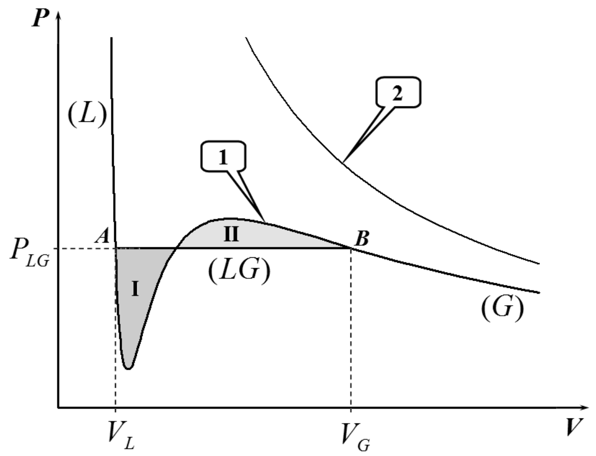


Figure 1. Van der Waals isotherm of gas/liquid (curve 1) and the isotherm of an ideal gas (curve 2).

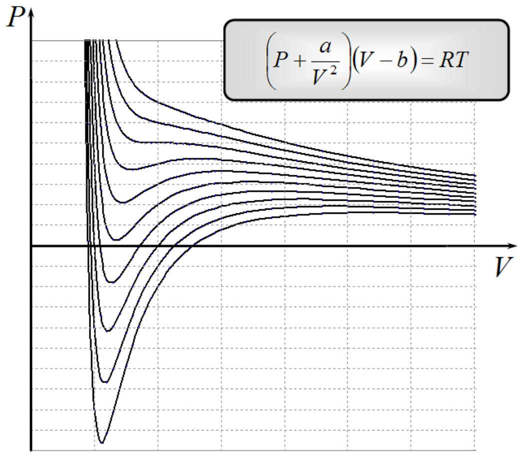


Figure 2. Several isotherms for van der Waals equation of state.

With increasing temperature the straight segment AB on the isotherm shrinks to a single point when the temperature and the pressure reaches some values T_c and $P_{LG}=P_c$, respectively. The parameters P_c and T_c are called critical and can be measured experimentally with high degree of accuracy.

A2	Express the van der Waals constants a and b in terms of T_c and P_c. **(1.3 points)**
A3	For water $T_c = 647$ K and $P_c = 2.2 \cdot 10^7$ Pa. Calculate a_w and b_w for water. **(0.2 points)**
A4	Estimate the diameter of water molecules d_w. **(0.2 points)**

Part B. Properties of gas and liquid (6 points)

This part of the problem deals with the properties of water in the gaseous and liquid states at temperature $T = 100\ °\text{C}$. The saturated vapor pressure at this temperature is known to be $p_{LG} = p_0 = 1.0 \cdot 10^5$ Pa, and the molar mass of water is $\mu = 1.8 \cdot 10^{-2}\ \frac{\text{kg}}{\text{mole}}$.

Gaseous state

It is reasonable to assume that the inequality $V_G \gg b$ is valid for the description of water properties in a gaseous state.

B1	Derive the formula for the volume V_G and express it in terms of R, T, p_0, and a. **(0.8 points)**

Almost the same volume V_{G0} can be approximately evaluated using the ideal gas law.

B2	Evaluate in percentage the relative decrease in the gas volume due to intermolecular forces, $\frac{\Delta V_G}{V_{G0}} = \frac{V_{G0} - V_G}{V_{G0}}$. **(0.3 points)**

If the system volume is reduced below V_G, the gas starts to condense. However, thoroughly purified gas can remain in a mechanically metastable state (called supercooled vapor) until its volume reaches a certain value $V_{G\min}$.

The condition of mechanical stability of supercooled gas at constant temperature is written as: $\frac{dP}{dV} < 0$.

B3	Find and evaluate how many times the volume of water vapor can be reduced and still remains in a metastable state. In other words, what is $V_G / V_{G\min}$? **(0.7 points)**

Liquid state

For the van der Waals' description of water in a liquid state it is reasonable to assume that the following inequality holds $P \ll a/V^2$.

B4	Express the volume of liquid water V_L in terms of a, b, R, and T. **(1.0 points)**

Assuming that $bRT \ll a$, find the following characteristics of water. *Do not be surprised if some of the data evaluated do not coincide with the well-known tabulated values!*

B5	Express the liquid water density ρ_L in some of the terms of μ, a, b, R and evaluate it. **(0.5 points)**
B6	Express the volume thermal expansion coefficient $\alpha = \frac{1}{V_L}\frac{\Delta V_L}{\Delta T}$ in terms of a, b, R, and evaluate it. **(0.6 points)**
B7	Express the specific heat of water vaporization L in terms of μ, a, b, R and evaluate it. **(1.1 points)**
B8	Considering the monomolecular layer of water, estimate the surface tension σ of water. **(1.2 points)**

Part C. Liquid-gas system (3 points)

From Maxwell's rule (equalities of areas, by applying trivial integration) and the van der Waals' equation of state together with the approximations made in Part B, it can be shown that the saturated vapor pressure p_{LG} depends on temperature T as follows

$$\ln p_{LG} = A + \frac{B}{T}, \tag{3}$$

where A and B are some constants, that can be expressed in terms of a and b as $A = \ln\left(\frac{a}{b^2}\right) - 1$; $B = -\frac{a}{bR}$

W. Thomson showed that the pressure of saturated vapor depends on the curvature of the liquid surface. Consider a liquid that does not wet the material of a capillary (contact angle 180°). When the capillary is immersed into the liquid, the liquid in the capillary drops to a certain level because of the surface tension (see Figure 3).

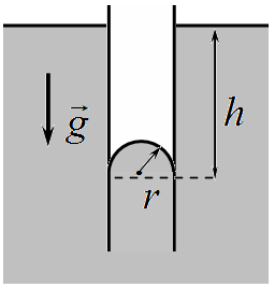


Figure 3. Capillary immersed in a liquid that does not wet its material

C1	Find a small change in pressure Δp_T of the saturated vapor over the curved surface of liquid and express it in terms of the vapor density ρ_s, the liquid density ρ_L, the surface tension σ and the radius of surface curvature r. **(1.3 points)**

Metastable states, considered in part B3, are widely used in real experimental setups, such as the cloud chamber designed for registration of elementary particles. They also occur in natural phenomena such as the formation of morning dew. Supercooled vapor is subject to condensation by forming liquid droplets. Very small droplets evaporate quickly but large enough ones can still grow.

C2	Suppose that at the evening temperature of $t_e = 20$ °C the water vapor in the air was saturated, but in the morning the ambient temperature has fallen by a small amount of $\Delta t = 5.0$ °C. Assuming that the vapor pressure has remained unchanged, estimate the minimum radius of droplets that can grow. Use the tabulated value of water surface tension $\sigma = 7.3 \cdot 10^{-2}\ N/m$. **(1.7 points)**

Problem3. Simplest model of gas discharge (10 points)

An electric current flowing through a gas is called a gas discharge. There are many types of gas discharges including glow discharge in lighting lamps, arc discharge inwelding and the well known spark discharge that occurs between the clouds and the earth in the form of lightning.

Part A. Non-self-sustained gas discharge (4.8points)

In this part of the problem the so-called non-self-sustained gas discharge is studied. To maintain it a permanent operationan external ionizer is needed, which creates Z_{ext} pairs of singly ionized ions and free electrons per unit volume and per unit time uniformly in the volume.

When an external ionizer is switched on, the number of electrons and ions starts to grow. Unlimited increase in the number densities of electrons and ions in the gas is prevented by the recombination process in which a free electron recombineswith an ion to form a neutral atom. The number of recombining eventsZ_{rec} that occurs in the gas per unit volume and per unit time is given by

$$Z_{\mathrm{rec}} = r n_e n_i,$$

where r is a constant called the recombination coefficient, and n_e,$n_i$denotethe electron and ion number densities, respectively.

Suppose that at time $t = 0$the external ionizer is switched on and the initial number densities of electrons and ions in the gas are both equal to zero. Then, the electron number density $n_e(t)$ depends on time$t$as follows:

$$n_e(t) = n_0 + a \tanh bt,$$

where n_0, a and $b$are some constants, and $\tanh x$stands for the hyperbolic tangent.

A1	Find$n_0, a, b$and express them in terms ofZ_{ext} and$r$.**(1.8points)**

Assume that there are two external ionizers available. When the first one is switched on, the electron number density in the gas reaches its equilibrium value of be $n_{e1} = 12 \cdot 10^{10}\,\mathrm{cm}^{-3}$. When the second external ionizer is switched on, the electron number density reaches its equilibrium value of $n_{e2} = 16 \cdot 10^{10}\,\mathrm{cm}^{-3}$.

A2	Find the electron number density$n_e$at equilibriumwhen both external ionizers are switched on simultaneously.**(0.6 points)**

Attention!*In what follows it is assumed that the external ionizer is switched on for quite long period of time such that all processes have become stationary and do not depend on time. Completely neglect the electric field due the charge carriers.*

Assume that the gas fills in the tube between the two parallel conductive plates of area S separated by the distance $L \ll \sqrt{S}$ from each other. The voltage U is applied across the plates to create an electric field between them. Assume that the number densities of both kinds of charge carriers remain almost constant along the tube.

Assumethat both the electrons (denoted by thesubscript$e$) and the ions (denoted by the subscript$i$) acquire the same ordered speed$v$due to the electric field strength$E$ found as

$$v = \beta E,$$

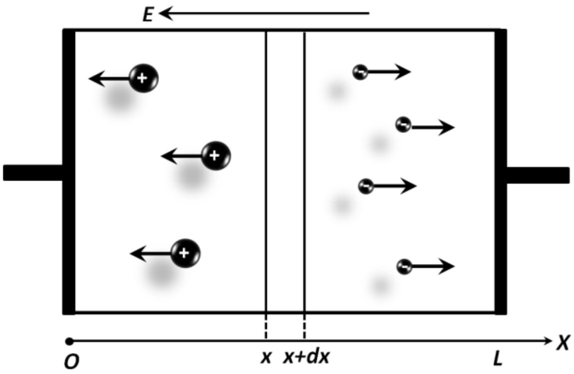
where $b$isa constant called charge mobility.

A3	Express the electric current$I$in the tube in terms of $U, \beta, L, S, Z_{\mathrm{ext}}, r$ and $e$whichis the elementary charge.**(1.7 points)**
A4	Find the resistivityρ_{gas} of the gas at sufficiently small values of the voltage applied and express it in terms of$\beta, L, Z_{\mathrm{ext}}, r$ and e.**(0.7 points)**

PartB. Self-sustained gas discharge (5.2 points)

In this part of the problemthe ignition of the self-sustained gas discharge is consideredto show how the electric current in the tube becomes self-maintaining.

Attention!*In the sequel assume that the external ionizer continues to operatewith the same Z_{ext} rate, neglect the electric field due to the charge carriers such that the electric field is uniform along the tube, and the recombination can be completely ignored.*



For the self-sustained gas discharge there are two important processes not considered above. The first process isasecondary electron emission, and the second one isa formation of electron avalanche. The secondary electron emission occurs when ions hit on the negative electrode, called a cathode, and the electrons are knocked out of it to move towards the positive electrode, called an anode. The ratio of the number of the knocked electrons$\dot{N}_e$per unit timeto the number of ions$\dot{N}_i$ hitting the cathode per unit time is called the coefficient of the secondary electron emission,$\gamma = \dot{N}_e/\dot{N}_i$.The formation of the electron avalanche isexplained as follows. The electric field accelerates free electrons which acquire enough kinetic energy to ionize the atoms in the gas by hitting them. As a result the number of free electrons moving towards the anodesignificantly increases. This process is described by the Townsend coefficient α, which characterizes an increase in the number of electronsdN_edue to moving N_e electrons that have passed the distancedl, i.e.

$$\frac{dN_e}{dl} = \alpha N_e.$$

The total current$I$ in any cross section of the gas tube consists of the ion $I_i(x)$and the electron$I_e(x)$ currents which, in the steady state, depend on the coordinate x, shown in the figureabove. The electron current $I_e(x)$ varies along the x-axis according to the formula

$$I_e(x) = C_1 e^{A_1 x} + A_2,$$

where A_1, A_2, C_1 are some constants.

B1	Find$A_1, A_2$and express them in terms of$Z_{\text{ext}}, \alpha, e, L, S$.**(2 points)**

The ion current $I_i(x)$ varies along the x-axis according to the formula

$$I_i(x) = C_2 + B_1 e^{B_2 x},$$

where B_1, B_2, C_2 are some constants.

B2	Find$B_1, B_2$and express them in terms of$Z_{\text{ext}}, \alpha, e, L, S, C_1$.**(0.6 points)**
B3	Write down the condition for $I_i(x)$at$x = L$.**(0.3 points)**
B4	Write down the condition for$I_i(x)$and$I_e(x)$at $x = 0$.**(0.6 points)**
B5	Find the total current$I$and express it in terms of$Z_{\text{ext}}, \alpha, \gamma, e, L, S$.Assume that it remains finite **(1.2 points)**

Let the Townsend coefficient αbe constant. When the length of the tube turns out greater than some critical value, i.e. $L > L_{\text{cr}}$, the external ionizer can be turned off and the discharge becomes self-sustained.

B6	Find$L_{cr}$and express it in terms of$Z_{\text{ext}}, \alpha, \gamma, e, L, S$.**(0.5 points)**