## Particles from the Sun[1] (Total Marks: 10)

Photons from the surface of the Sun and neutrinos from its core can tell us about solar temperatures and also confirm that the Sun shines because of nuclear reactions.

Throughout this problem, take the mass of the Sun to be $M_\odot = 2.00 \times 10^{30}$ kg, its radius, $R_\odot = 7.00 \times 10^8$ m, its luminosity (radiation energy emitted per unit time), $L_\odot = 3.85 \times 10^{26}$ W, and the Earth-Sun distance, $d_\odot = 1.50 \times 10^{11}$ m.

Note:

(i) $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + \text{ constant}$

(ii) $\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} + \text{ constant}$

(iii) $\int x^3e^{ax}dx = \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)e^{ax} + \text{ constant}$

### A Radiation from the sun :

| | | |
|---|---|---|
| A1 | Assume that the Sun radiates like a perfect blackbody. Use this fact to calculate the temperature, $T_s$, of the solar surface. | **0.3** |

The spectrum of solar radiation can be approximated well by the Wien distribution law. Accordingly, the solar energy incident on any surface on the Earth per unit time per unit frequency interval, $u(\nu)$, is given by

$$u(\nu) = A\frac{R_\odot^2}{d_\odot^2}\frac{2\pi h}{c^2}\nu^3 \exp(-h\nu/k_B T_s),$$

where $\nu$ is the frequency and $A$ is the area of the surface normal to the direction of the incident radiation.

Now, consider a solar cell which consists of a thin disc of semiconducting material of area, $A$, placed perpendicular to the direction of the Sun's rays.

| | | |
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| A2 | Using the Wien approximation, express the total radiated solar power, $P_{in}$, incident on the surface of the solar cell, in terms of $A$, $R_\odot$, $d_\odot$, $T_s$ and the fundamental constants $c$, $h$, $k_B$. | **0.3** |
| A3 | Express the number of photons, $n_\gamma(\nu)$, per unit time per unit frequency interval incident on the surface of the solar cell in terms of $A$, $R_\odot$, $d_\odot$, $T_s$, $\nu$ and the fundamental constants $c$, $h$, $k_B$. | **0.2** |

The semiconducting material of the solar cell has a "band gap" of energy, $E_g$. We assume the following model. Every photon of energy $E \geq E_g$ excites an electron across the band gap. This electron contributes an energy, $E_g$, as the useful output energy, and any extra energy is dissipated as heat (not converted to useful energy).

| | | |
|---|---|---|
| A4 | Define $x_g = h\nu_g/k_B T_s$ where $E_g = h\nu_g$. Express the useful output power of the cell, $P_{out}$, in terms of $x_g$, $A$, $R_\odot$, $d_\odot$, $T_s$ and the fundamental constants $c$, $h$, $k_B$. | **1.0** |
| A5 | Express the efficiency, $\eta$, of this solar cell in terms of $x_g$. | **0.2** |
| A6 | Make a qualitative sketch of $\eta$ versus $x_g$. The values at $x_g = 0$ and $x_g \to \infty$ should be clearly shown. What is the slope of $\eta(x_g)$ at $x_g = 0$ and $x_g \to \infty$? | **1.0** |
| A7 | Let $x_0$ be the value of $x_g$ for which $\eta$ is maximum. Obtain the cubic equation that gives $x_0$. Estimate the value of $x_0$ within an accuracy of $\pm 0.25$. Hence calculate $\eta(x_0)$. | **1.0** |
| A8 | The band gap of pure silicon is $E_g = 1.11$ eV. Calculate the efficiency, $\eta_{Si}$, of a silicon solar cell using this value. | **0.2** |

[1] *Amol Dighe (TIFR), Anwesh Mazumdar (HBCSE-TIFR) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

In the late nineteenth century, Kelvin and Helmholtz (KH) proposed a hypothesis to explain how the Sun shines. They postulated that starting as a very large cloud of matter of mass, $M_\odot$, and negligible density, the Sun has been shrinking continuously. The shining of the Sun would then be due to the release of gravitational potential energy through this slow contraction.

| | | |
|---|---|---|
| A9 | Let us assume that the density of matter is uniform inside the Sun. Find the total gravitational potential energy, $\Omega$, of the Sun at present, in terms of $G$, $M_\odot$ and $R_\odot$. | **0.3** |
| A10 | Estimate the maximum possible time, $\tau_{\mathrm{KH}}$ (in years), for which the Sun could have been shining, according to the KH hypothesis. Assume that the luminosity of the Sun has been constant throughout this period. | **0.5** |

The $\tau_{\mathrm{KH}}$ calculated above does not match the age of the solar system estimated from studies of meteorites. This shows that the energy source of the Sun cannot be purely gravitational.

**B Neutrinos from the Sun :**

In 1938, Hans Bethe proposed that nuclear fusion of hydrogen into helium in the core of the Sun is the source of its energy. The net nuclear reaction is:

$$4\ {}^{1}\mathrm{H} \longrightarrow {}^{4}\mathrm{He} + 2\mathrm{e}^{+} + 2\nu_{\mathrm{e}}$$

The “electron neutrinos”, $\nu_{\mathrm{e}}$, produced in this reaction may be taken to be massless. They escape the Sun and their detection on the Earth confirms the occurrence of nuclear reactions inside the Sun. Energy carried away by the neutrinos can be neglected in this problem.

| | | |
|---|---|---|
| B1 | Calculate the flux density, $\Phi_\nu$, of the number of neutrinos arriving at the Earth, in units of $\mathrm{m}^{-2}\mathrm{s}^{-1}$. The energy released in the above reaction is $\Delta E = 4.0 \times 10^{-12}\mathrm{J}$ . Assume that the energy radiated by the Sun is entirely due to this reaction. | **0.6** |

Travelling from the core of the Sun to the Earth, some of the electron neutrinos, $\nu_e$, are converted to other types of neutrinos, $\nu_{\mathrm{x}}$. The efficiency of the detector for detecting $\nu_{\mathrm{x}}$ is 1/6 of its efficiency for detecting $\nu_{\mathrm{e}}$. If there is no neutrino conversion, we expect to detect an average of $N_1$ neutrinos in a year. However, due to the conversion, an average of $N_2$ neutrinos ($\nu_{\mathrm{e}}$ and $\nu_{\mathrm{x}}$ combined) are actually detected per year.

| | | |
|---|---|---|
| B2 | In terms of $N_1$ and $N_2$, calculate what fraction, $f$, of $\nu_{\mathrm{e}}$ is converted to $\nu_{\mathrm{x}}$. | **0.4** |

In order to detect neutrinos, large detectors filled with water are constructed. Although the interactions of neutrinos with matter are very rare, occasionally they knock out electrons from water molecules in the detector. These energetic electrons move through water at high speeds, emitting electromagnetic radiation in the process. As long as the speed of such an electron is greater than the speed of light in water (refractive index, $n$), this radiation, called Cherenkov radiation, is emitted in the shape of a cone.

| | | |
|---|---|---|
| B3 | Assume that an electron knocked out by a neutrino loses energy at a constant rate of $\alpha$ per unit time, while it travels through water. If this electron emits Cherenkov radiation for a time, $\Delta t$, determine the energy imparted to this electron ($E_{\mathrm{imparted}}$) by the neutrino, in terms of $\alpha$, $\Delta t$, $n$, $m_{\mathrm{e}}$ and $c$. (Assume the electron to be at rest before its interaction with the neutrino.) | **2.0** |

The fusion of H into He inside the Sun takes place in several steps. Nucleus of ${}^{7}\mathrm{Be}$ (rest mass, $m_{\mathrm{Be}}$) is produced in one of these intermediate steps. Subsequently, it can absorb an electron, producing a ${}^{7}\mathrm{Li}$ nucleus (rest mass, $m_{\mathrm{Li}} < m_{\mathrm{Be}}$) and emitting a $\nu_{\mathrm{e}}$. The corresponding nuclear reaction is:

$${}^{7}\mathrm{Be} + \mathrm{e}^{-} \longrightarrow {}^{7}\mathrm{Li} + \nu_{\mathrm{e}}$$

When a Be nucleus ($m_{\mathrm{Be}} = 11.65 \times 10^{-27}$ kg) is at rest and absorbs an electron also at rest, the emitted neutrino has energy $E_\nu = 1.44 \times 10^{-13}$ J. However, the Be nuclei are in random thermal motion due to the temperature $T_{\mathrm{c}}$ at the core of the Sun, and act as moving neutrino sources. As a result, the energy of emitted neutrinos fluctuates with a root mean square (rms) value $\Delta E_{rms}$.

| | | |
|---|---|---|
| B4 | If $\Delta E_{rms}$=$5.54 \times 10^{-17}$J, calculate the rms speed of the Be nuclei, $V_{\mathrm{Be}}$, and hence estimate $T_{\mathrm{c}}$. (Hint: $\Delta E_{rms}$ depends on the rms value of the component of velocity along the line of sight). | **2.0** |

**The Extremum Principle**[1] **(Total Marks: 10)**

## A The Extremum Principle in Mechanics

Consider a horizontal frictionless $x-y$ plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass $m$ in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed $v_1$ along a line making an angle $\theta_1$ with the $x$-axis. It reaches point P in region II traveling with speed $v_2$ along a line that makes an angle $\theta_2$ with the $x$-axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

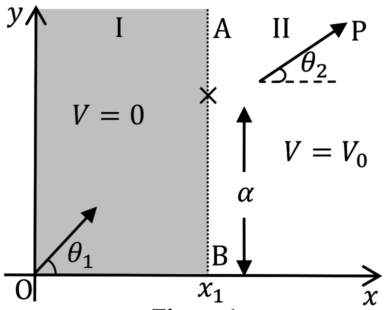



**Figure 1**

| | | |
|---|---|---|
| A1 | Obtain an expression for $v_2$ in terms of $m$, $v_1$ and $V_0$. | **0.2** |
| A2 | Express $v_2$ in terms of $v_1, \theta_1$ and $\theta_2$. | **0.3** |

We define a quantity called action $A = m\int v(s)ds$, where $ds$ is the infinitesimal length along the trajectory of a particle of mass $m$ moving with speed $v(s)$. The integral is taken over the path. As an example, for a particle moving with constant speed $v$ on a circular path of radius $R$, the action $A$ for one revolution will be $2\pi mRv$. For a particle with constant energy $E$, it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which $A$ defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

| | | |
|---|---|---|
| A3 | PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates $(0,0)$ and $(x_0, y_0)$ respectively and the boundary point where the particle transits from region I to region II have coordinates $(x_1, \alpha)$. Note that $x_1$ is fixed and the action depends on the coordinate $\alpha$ only. State the expression for the action $A(\alpha)$. Use PLA to obtain the relationship between $v_1/v_2$ and these coordinates. | **1.0** |

## B The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices $n_1$ and $n_2$ respectively. The two media are separated by a line parallel to the $x$-axis. The light ray makes an angle $i_1$ with the $y$-axis in medium I and $i_2$ in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

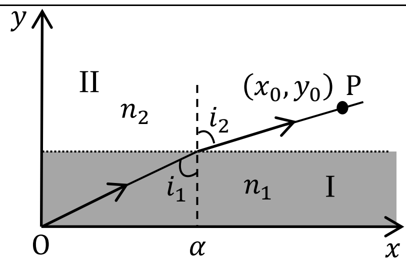



**Figure 2**

| | | |
|---|---|---|
| B1 | The principle states that between two fixed points, a light ray moves along a path such that time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle. | **0.5** |

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

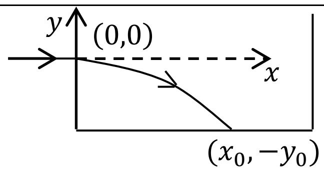



**Figure 3: Tank of Sugar Solution**

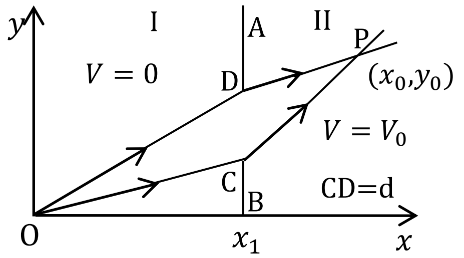
| | | |
|---|---|---|
| B2 | Assume that the refractive index $n(y)$ depends only on $y$. Use the equation obtained in B1 to obtain the expression for the slope $dy/dx$ of the beam's path in terms of refractive index $n_0$ at $y = 0$ and $n(y)$. | **1.5** |
| B3 | The laser beam is directed horizontally from the origin $(0,0)$ into the sugar solution at a height $y_0$ from the bottom of the tank as shown in figure 3. Take $n(y) = n_0 - ky$ where $n_0$ and $k$ are positive constants. Obtain an expression for $x$ in terms of $y$ and related quantities for the actual trajectory of the laser beam. | **1.2** |

[1] *Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

| | | |
|---|---|---|
| | You may use: $\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta) + \text{constant}$, where $\sec\theta = 1/\cos\theta$ or $\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x + \sqrt{x^2-1}\right) + \text{constant}$ | |
| B4 | Obtain the value of $x_0$, the point where the beam meets the bottom of the tank. Take $y_0 = 10.0$ cm, $n_0 = 1.50$, $k = 0.050\ \text{cm}^{-1}$ (1 cm = $10^{-2}$ m). | **0.8** |

**C The Extremum Principle and the Wave Nature of Matter**

We now explore the connection between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

| | | |
|---|---|---|
| C1 | As the particle moves along its trajectory by an infinitesimal distance $\Delta s$, relate the change $\Delta\varphi$ in the phase of its de Broglie wave to the change $\Delta A$ in the action and the Planck constant. | **0.6** |
| C2 | Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width $d$ in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\varphi_{CD}$ between the two paths to first order.   **Figure 4** | **1.2** |

**D Matter Wave Interference**

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition $A_1B_1$ at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7\ \text{m s}^{-1}$ and $\theta = 10.0000^\circ$. The potential in II is such that speed $v_2 = 1.9900 \times 10^7\ \text{m s}^{-1}$. The distance $x_0 - x_1$ is 250.00 mm (1mm $= 10^{-3}$m). Ignore electron-electron interaction.

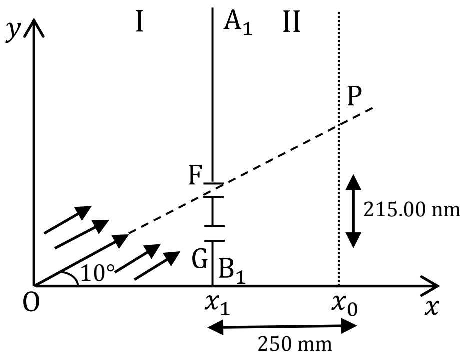


**Figure 5**

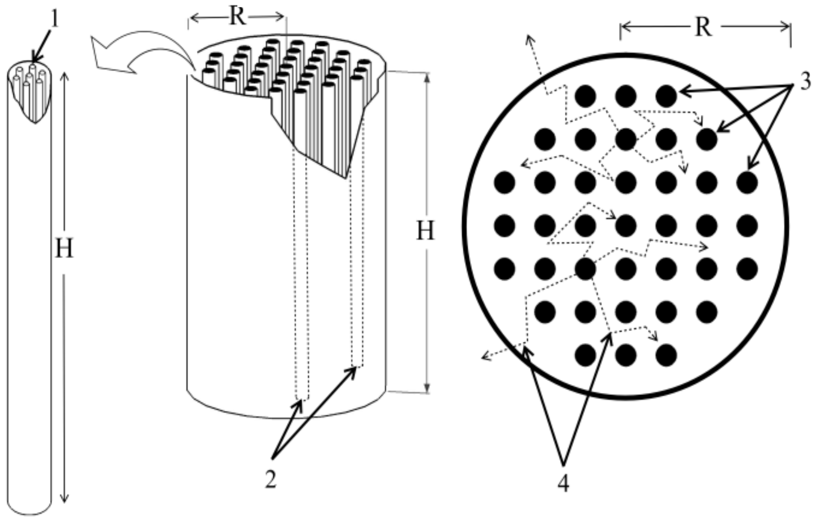
| | | |
|---|---|---|
| D1 | If the electrons at O have been accelerated from rest, calculate the accelerating potential $U_1$. | **0.3** |
| D2 | Another identical slit G is made in the partition $A_1B_1$ at a distance of 215.00 nm (1nm $= 10^{-9}$m) below slit F (Fig. 5). If the phase difference between de Broglie waves arriving at P through the slits F and G is $2\pi\beta$, calculate $\beta$. | **0.8** |
| D3 | What is the smallest distance $\Delta y$ from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$ useful] | **1.2** |
| D4 | The beam has a square cross section of 500nm × 500nm and the setup is 2 m long. What should be the minimum flux density $I_{min}$ (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time? | **0.4** |

**The Design of a Nuclear Reactor**[1] **(Total Marks: 10)**

Uranium occurs in nature as $UO_2$ with only 0.720% of the uranium atoms being $^{235}U$. Neutron induced fission occurs readily in $^{235}U$ with the emission of 2-3 fission neutrons having high kinetic energy. This fission probability will increase if the neutrons inducing fission have low kinetic energy. So by reducing the kinetic energy of the fission neutrons, one can induce a chain of fissions in other $^{235}U$ nuclei. This forms the basis of the power generating nuclear reactor (NR).

A typical NR consists of a cylindrical tank of height $H$ and radius $R$ filled with a material called moderator. Cylindrical tubes, called fuel channels, each containing a cluster of cylindrical fuel pins of natural $UO_2$ in solid form of height $H$, are kept axially in a square array. Fission neutrons, coming outward from a fuel channel, collide with the moderator, losing energy and reach the surrounding fuel channels with low enough energy to cause fission (Figs I-III). Heat generated from fission in the pin is transmitted to a coolant fluid flowing along its length. In the current problem we shall study some of the physics behind the (A) Fuel Pin, (B) Moderator and (C) NR of cylindrical geometry.

Fig-I Fig-II Fig-III




*Schematic sketch of the Nuclear Reactor* (NR)
Fig-I: Enlarged view of a fuel channel (1-*Fuel Pins*)
Fig-II: A view of the NR (2-*Fuel Channels*)
Fig-III: Top view of NR (3-*Square Arrangement of Fuel Channels* and 4-*Typical Neutron Paths*).
Only components relevant to the problem are shown (e.g. control rods and coolant are not shown).

## A Fuel Pin

| Data for $UO_2$ | | |
|---|---|---|
| | 1. Molecular weight $M_w = 0.270$ kg mol$^{-1}$ | 2. Density $\rho = 1.060\times10^4$ kg m$^{-3}$ |
| | 3. Melting point $T_m = 3.138\times10^3$ K | 4. Thermal conductivity $\lambda = 3.280$ W m$^{-1}$ K$^{-1}$ |

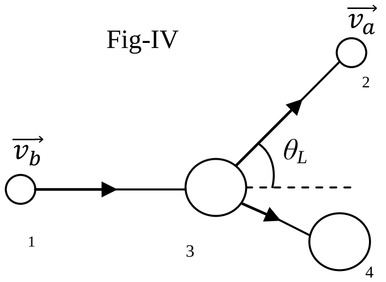
| | | |
|---|---|---|
| A1 | Consider the following fission reaction of a stationary $^{235}U$ after it absorbs a neutron of negligible kinetic energy. $$^{235}U + {}^{1}n \rightarrow {}^{94}Zr + {}^{140}Ce + 2\,{}^{1}n + \Delta E$$ Estimate $\Delta E$ (in MeV) the total fission energy released. The nuclear masses are: $m(^{235}U) = 235.044$ u; $m(^{94}Zr) = 93.9063$ u; $m(^{140}Ce) = 139.905$ u; $m(^{1}n) = 1.00867$ u and 1 u = 931.502 MeV c$^{-2}$. Ignore charge imbalance. | **0.8** |
| A2 | Estimate $N$ the number of $^{235}U$ atoms per unit volume in natural $UO_2$. | **0.5** |
| A3 | Assume that the neutron flux density, $\varphi = 2.000\times10^{18}$ m$^{-2}$ s$^{-1}$ on the fuel is uniform. The fission cross-section (effective area of the target nucleus) of a $^{235}U$ nucleus is $\sigma_f = 5.400\times10^{-26}$ m$^2$. If 80.00% of the fission energy is available as heat, estimate $Q$ (in W m$^{-3}$), the rate of heat production in the pin per unit volume. 1MeV = $1.602\times10^{-13}$ J | **1.2** |
| A4 | The steady-state temperature difference between the center ($T_c$) and the surface ($T_s$) of the pin can be expressed as $T_c - T_s = k\,F(Q,a,\lambda)$, where $k = 1/4$ is a dimensionless constant and $a$ is the radius of the pin. Obtain $F(Q,a,\lambda)$ by dimensional analysis. Note that $\lambda$ is the thermal conductivity of $UO_2$. | **0.5** |

[1] *Joseph Amal Nathan (BARC) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

| | | |
|---|---|---|
| A5 | The desired temperature of the coolant is $5.770\times10^{2}$ K. Estimate the upper limit $a_u$ on the radius $a$ of the pin. | **1.0** |

**B The Moderator**

Consider the two dimensional elastic collision between a neutron of mass 1 u and a moderator atom of mass $A$ u. Before collision all the moderator atoms are considered at rest in the laboratory frame (LF). Let $\vec{v_b}$ and $\vec{v_a}$ be the velocities of the neutron before and after collision respectively in the LF. Let $\vec{v_m}$ be the velocity of the center of mass (CM) frame relative to LF and $\theta$ be the neutron scattering angle in the CM frame. All the particles involved in collisions are moving at nonrelativistic speeds.

| | | |
|---|---|---|
| B1 | The collision in LF is shown schematically, where $\theta_L$ is the scattering angle (Fig-IV). Sketch the collision schematically in CM frame. Label the particle velocities for 1, 2 and 3 in terms of $\vec{v_b}$, $\vec{v_a}$ and $\vec{v_m}$. Indicate the scattering angle $\theta$.<br>Fig-IV<br><br>*Collision in the Laboratory Frame*<br>1-*Neutron* before collision<br>2-*Neutron* after collision<br>3-*Moderator Atom* before collision<br>4-*Moderator Atom* after collision | **1.0** |
| B2 | Obtain $v$ and $V$, the speeds of the neutron and moderator atom in the CM frame after collision, in terms of $A$ and $v_b$. | **1.0** |
| B3 | Derive an expression for $G(\alpha, \theta) = E_a/E_b$ , where $E_b$ and $E_a$ are the kinetic energies of the neutron, in the LF, before and after the collision respectively and $\alpha \equiv [(A-1)\ /\ (A+1)]^2$. | **1.0** |
| B4 | Assume that the above expression holds for $D_2O$ molecule. Calculate the maximum possible fractional energy loss $f_l \equiv \frac{E_b - E_a}{E_b}$ of the neutron for the $D_2O$ (20 u) moderator. | **0.5** |

**C The Nuclear Reactor**

To operate the NR at any constant neutron flux $\psi$ (steady state), the leakage of neutrons has to be compensated by an excess production of neutrons in the reactor. For a reactor in cylindrical geometry the leakage rate is $k_1\,[(2.405/R)^2 + (\pi/H)^2]\,\psi$ and the excess production rate is $k_2\,\psi$. The constants $k_1$ and $k_2$ depend on the material properties of the NR.

| | | |
|---|---|---|
| C1 | Consider a NR with $k_1 = 1.021\times10^{-2}$ m and $k_2 = 8.787\times10^{-3}$ m$^{-1}$. Noting that for a fixed volume the leakage rate is to be minimized for efficient fuel utilization, obtain the dimensions of the NR in the steady state. | **1.5** |
| C2 | The fuel channels are in a square arrangement (Fig-III) with the nearest neighbour distance 0.286 m. The effective radius of a fuel channel (if it were solid) is $3.617\times10^{-2}$ m. Estimate the number of fuel channels $F_n$ in the reactor and the mass $M$ of $UO_2$ required to operate the NR in steady state. | **1.0** |