Zero-length springs and slinky coils

A zero effective length spring (ZLS) is a spring for which the force is proportional to the spring's length, $F = kL$ for $L > L_0$ where L_0 is the minimal length of the spring as well as its unstretched length. Figure 1 shows the relation between the force F and the spring length L for a ZLS, where the slope of the line is the spring constant k.

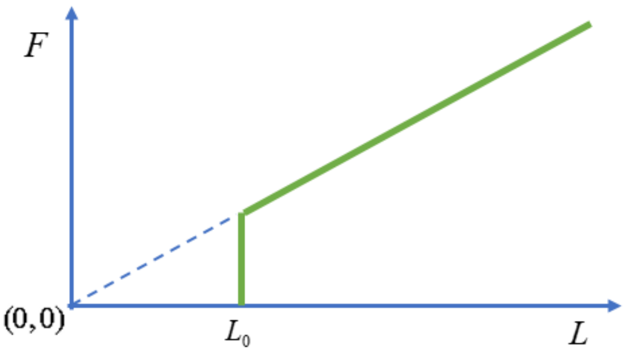

Figure 1: the relation between the force F and the spring length L

A ZLS is useful in seismography and allows very accurate measurement of changes in the gravitational acceleration g. Here, we shall consider a homogenous ZLS, whose weight Mg exceeds kL_0. We define a corresponding dimensionless ratio, $\alpha = kL_0/Mg < 1$ to characterize the relative softness of the spring. The toy known as “slinky” may be (but not necessarily) such a ZLS.

Part A: Statics (3.0 points)

A.1	Consider a segment of length $\Delta\ell$ of the unstretched ZLS spring which is then stretched by a force F, under weightless conditions. What is the length Δy of this segment as a function of $F, \Delta\ell$ and the parameters of the spring?	0.5pt

A.2	For a segment of length $\Delta\ell$, calculate the work ΔW required to stretch it from its original length $\Delta\ell$ to a length Δy.	0.5pt

Throughout this question, we will denote a point on the spring by its distance $0 \le \ell \le L_0$ from the bottom of the spring when it is unstretched. In particular, for every point on the spring, ℓ remains unchanged as the spring stretches.

A.3	Suppose that we hang the spring by its top end, so that it stretches under its own weight. What is the total length H of the suspended spring in equilibrium? Express your answers in terms of L_0 and α.	2.0pt

Part B: Dynamics (5.5 points)

Experiments show that when the spring is hung at rest and then released, it gradually contracts from the top, while the lower part remains stationary (see Figure 2). As time advances, the contracting part moves as a solid chunk and accumulates additional turns of the spring, while the stationary part becomes shorter. Every point on the spring begins to move only when the moving part reaches it. The bottom end of the spring starts moving only when the spring is fully collapsed and reaches its unstretched length L_0. After that, the contracted spring continues falling straight downwards, without tumbling, as a rigid body under the influence of gravity.

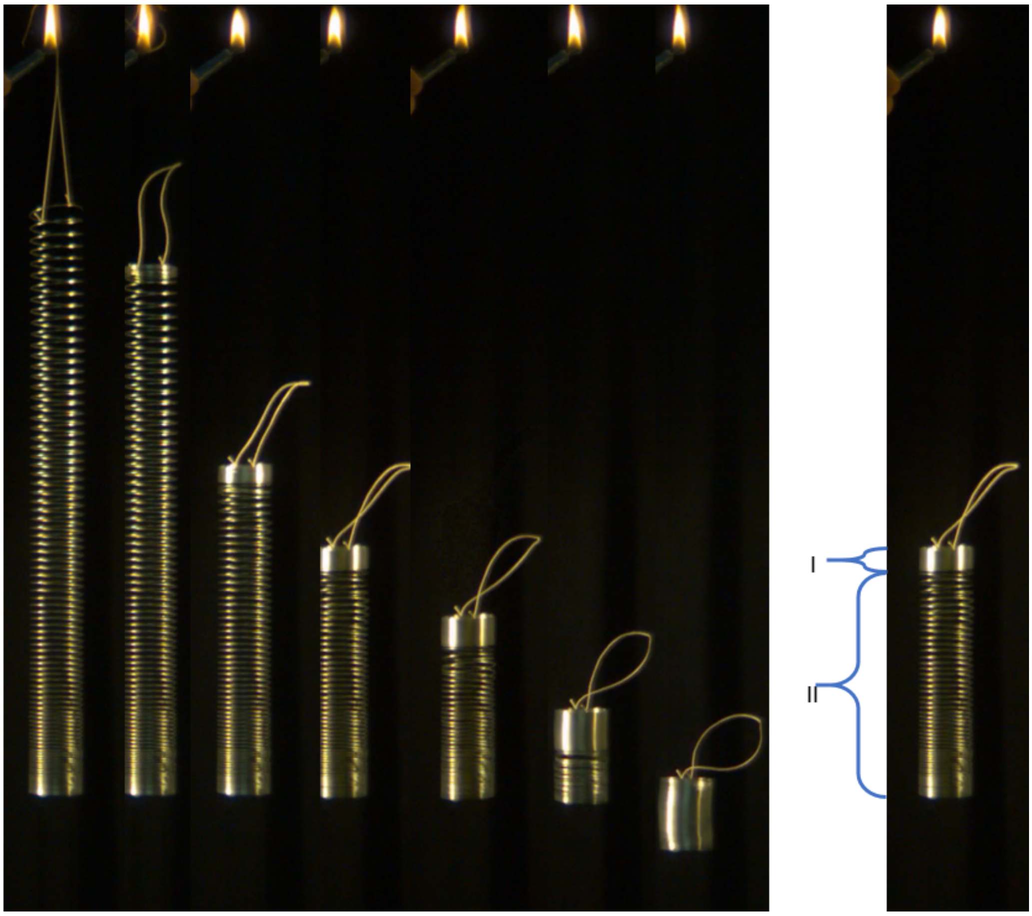

Figure 2: Left: a sequence of pictures taken during the free fall of slinky. Right: the moving part I and the stationary part II during the free fall of the spring.

In the remaining parts of the question, you are asked to base your solution on this described model. You may neglect air resistance, but you are not allowed to neglect L_0.

B.1	Calculate the time t_c it takes from the moment the spring is released, until it fully collapses back to its minimal length L_0. Express your answer in terms of L_0, g and α. Compute the numerical value of t_c for a spring with $k = 1.02\ \mathrm{N/m}$, $L_0 = 0.055\,\mathrm{m}$ and $M = 0.201\,\mathrm{kg}$, while taking g to be $9.80\,\mathrm{m/s^2}$.	2.5pt

B.2	In this task ℓ is used to denote the coordinate of the boundary between parts I (in figure 2, the moving part) and II (the stationary part). At a certain moment, while a stationary part still exists its mass is $m(\ell) = \frac{\ell}{L_0} M$, and the moving part moves with uniform instantaneous velocity $v_I(\ell)$. Show that at this moment (while there exists a stationary part) the velocity of the moving part is $v_I(\ell) = \sqrt{A\ell + B}$. Express the constants A and B in terms of L_0, g and α.	2.5pt

B.3	Based on B.2, find the minimum speed $v_{\min}$ of the moving part of the spring in the course of its motion, after its release and before it hits the ground. Express your answer in terms of L_0, α, A and B.	0.5pt

Part C: Energetics (1.5 points)

C.1	Calculate the amount of mechanical energy Q that was lost by generating heat, from the moment the spring is released until just before the spring hits the ground. Express your answer in terms of L_0, M, g and α.	1.5pt

The Physics of a Microwave Oven

This question discusses the generation of microwave radiation in a microwave oven, and its use to heat up food. The microwave radiation is generated in a device called "magnetron". Part A concerns the operation of the magnetron, while part B deals with the absorption of microwave radiation in food.

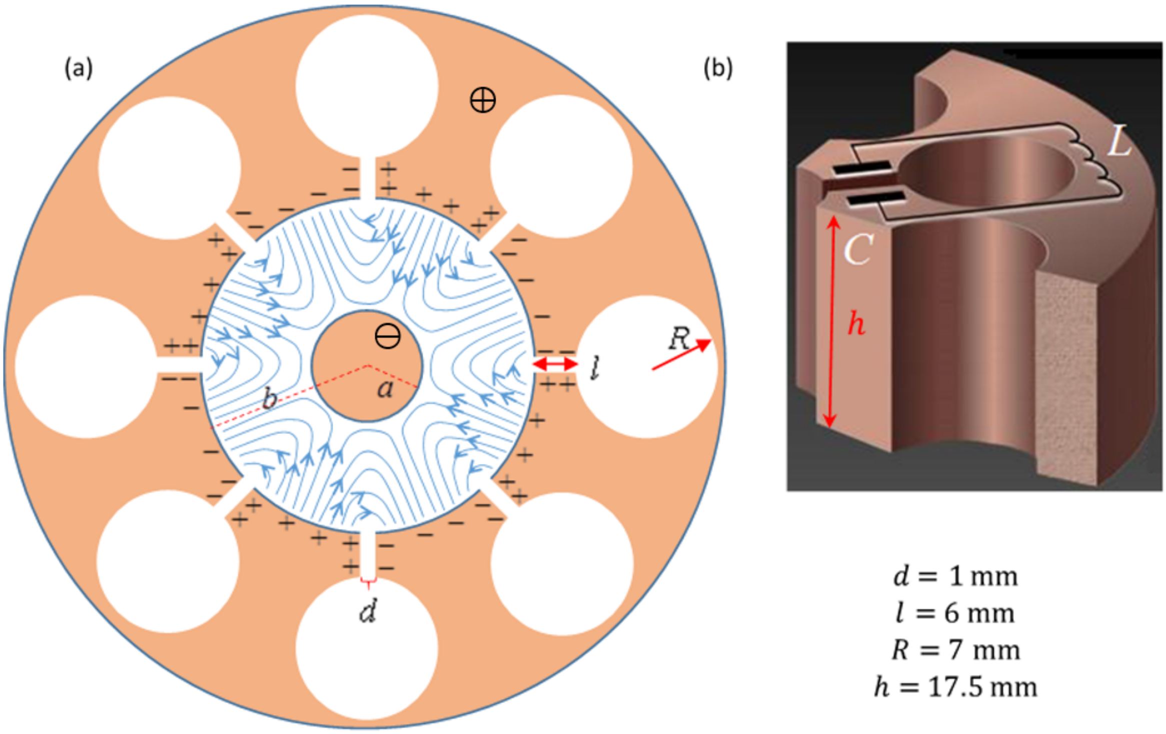

Figure 1

Part A: The structure and operation of a magnetron (6.6 points)

A magnetron is a device for the generation of microwave radiation, either in pulses (for radar applications), or continuously (e.g., in a microwave oven). The magnetron has a mode of self-amplifying oscillations. Supplying the magnetron with static (non-alternating) voltage quickly excites this mode. The microwave radiation thus created is transmitted out of the magnetron.

A typical microwave oven magnetron consists of a solid copper cylindrical cathode (with radius a) and a surrounding anode (with radius b). The latter has the shape of a thick cylindrical shell into which cylindrical cavities are drilled. These cavities are known as "resonators". One of the resonators is coupled to an antenna which will transmit the microwave energy out; we will ignore the antenna in the following. All internal spaces are in vacuum. We will consider a typical magnetron with eight resonators, as depicted in Figure 1(a). The three-dimensional structure of a single resonator is shown in Figure 1(b). As indicated there, each of the eight cavities behaves as an inductor-capacitor (LC) resonator, with operating frequency $f = 2.45\ \mathrm{GHz}$.

A static uniform magnetic field is applied along the magnetron's longitudinal axis, pointing out of the page in Figure 1(a). In addition, a constant voltage is applied between the anode (positive potential) and the cathode (negative potential). Electrons emitted from the cathode reach the anode and charge it, such that they excite an oscillation mode in which the sign of the charge is opposite between every two adjacent resonators. The oscillation of the cavities amplify these oscillations.

The process described above creates an alternating electric field with the aforementioned frequency $f = 2.45\,\mathrm{GHz}$ (blue lines in Figure 1(a); the static field is not plotted) in the space between the cathode and the anode, in addition to the static field caused by the applied constant voltage. In the steady state, the typical amplitude of the alternating electric field between the anode and the cathode is approximately $\frac{1}{3}$ of the static electric field there. The electron motion in the space between the cathode and the anode is affected by both the static and the alternating parts of the field. This causes electrons that reach the anode to transfer about 80% of the energy they acquire from the static field into the alternating field. A minority of the ejected electrons returns to the cathode and releases additional electrons, further amplifying the alternating field.

Each resonator can be thought of as a capacitor and an inductor, see Figure 1(b). The capacitance mainly arises from the planar parts of the resonator surface, while the inductance stems from the cylindrical part. Assume that the current in the resonator flows uniformly very close to the surface of its cylindrical cavity, and that the strength of the magnetic field generated by this current is 0.6 times that of an ideal infinite solenoid. The various lengths defining the resonator geometry are given in Figure 1(b). The vacuum permittivity and permeability are $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}$ and $\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{H}}{\mathrm{m}}$, respectively.

A.1	Use the above data to estimate the frequency f_{est} of a single resonator. (Your result may differ from the actual value, $f = 2.45\,\mathrm{GHz}$. Use the **actual** value in the remainder of the question.)	0.4pt

Task A.2 below does not deal with the magnetron itself, but helps to introduce some of the relevant physics. Consider an electron moving in free space under the influence of a uniform electric field directed along the negative y axis, $\vec{E} = -E_0\hat{y}$, and a uniform magnetic field directed along the positive z axis, $\vec{B} = B_0\hat{z}$ (E_0 and B_0 are positive; $\hat{x}, \hat{y}, \hat{z}$ are unit vectors oriented in the conventional manner). Let us denote the electron velocity at time t by $\vec{u}(t)$. The drift velocity $\vec{u}_D$ of the electron is defined as its average velocity. We denote by m and $-e$ the mass and charge of the electron, respectively.

A.2	In each of the following two cases, find $\vec{u}_D$. In addition, draw in the Answer Sheet the electron's trajectory (in the lab frame) during the time interval $0 < t < \frac{4\pi m}{eB_0}$ if: 1. at $t = 0$ the electron velocity is $\vec{u}(0) = (3E_0/B_0)\hat{x}$, 2. at $t = 0$ the electron velocity is $\vec{u}(0) = -(3E_0/B_0)\hat{x}$.	1.5pt

We now resume our discussion of the magnetron. The distance between the cathode and the anode is $15\,\mathrm{mm}$. Assume that, due to the aforementioned energy loss to the alternating fields, the maximal kinetic energy of each electron does not exceed $K_{\mathrm{max}} = 800\,\mathrm{eV}$. The static magnetic field strength is $B_0 = 0.3\,\mathrm{T}$. The electron mass and charge are $m = 9.1 \cdot 10^{-31}\,\mathrm{kg}$ and $-e = -1.6 \cdot 10^{-19}\,\mathrm{C}$, respectively.

A.3	Numerically estimate the maximal radius r of the electron motion trajectory in the reference frame in which this motion is approximately circular, considering this reference frame as approximately inertial.	0.4pt

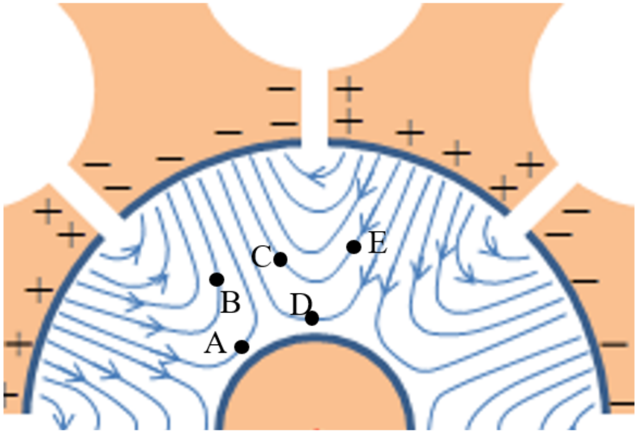

Figure 2

A.4	Figure 2 depicts the alternating electric field lines between the anode and the cathode at a given moment in time (the static field is not plotted). Indicate in the Answer Sheet which of the electrons positioned at A,B,C,D and E will drift towards the anode, which will drift towards the cathode and which will drift at a direction perpendicular to the radius at that moment.	1.2pt

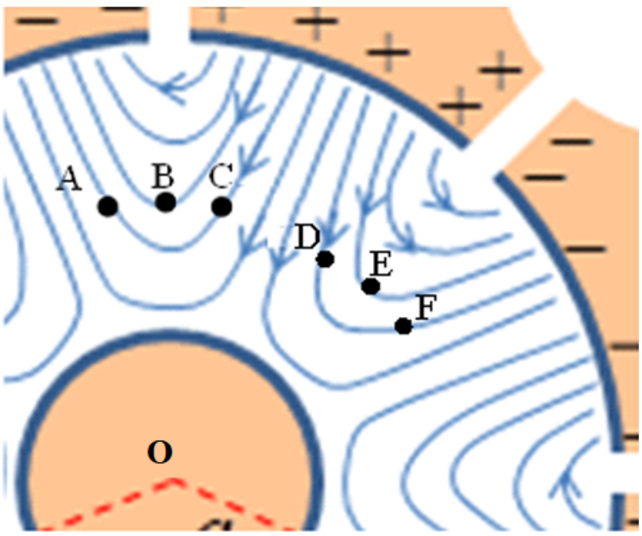

Figure 3

Figure 3 depicts the alternating electric field lines between the anode and the cathode (the static field is not plotted) at a given moment in time. The positions of six electrons at that moment are denoted by A, B, C, D, E and F. All electrons are at the same distance from the cathode.

A.5	Consider the situation shown in Figure 3. For each of the six electron pairs AB, AC, BC, DE, DF, EF, indicate in the Answer Sheet whether their drift will cause the angle between their position vectors (measured from the cathode's center O) to increase or decrease at that moment.	1.2pt

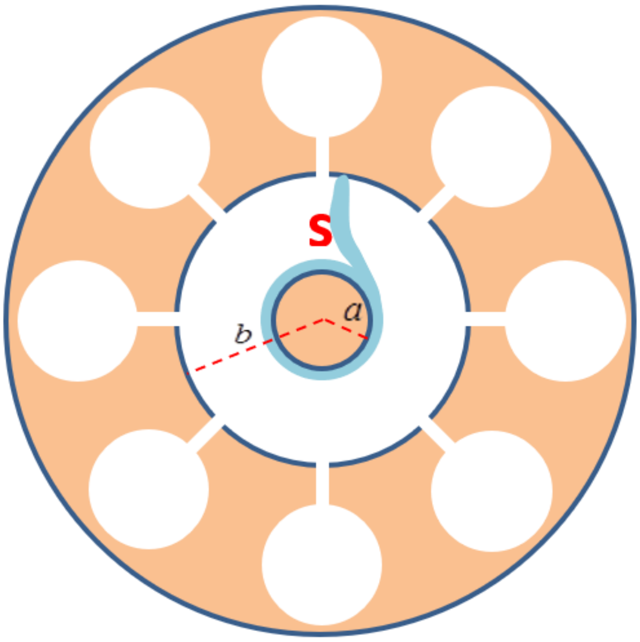

Figure 4

The pattern you have discovered in Task A.5 acts as a focusing mechanism, concentrating the electrons in the space between the cathode and anode into spokes. Figure 4 depicts one such spoke, denoted by S.

A.6	Depict in the Answer Sheet the other spokes at that moment. Indicate by arrows their direction of rotation, and calculate their average angular velocity ω_s.	0.8pt

Make the approximation that the total electric field half-way between the cathode and the anode is equal to its average static value along a radial line from the cathode to the anode, and that the spokes are approximately radial in that region. The cathode and anode radii (a and b, respectively) are defined in Figure 4.

A.7	Find an approximate expression for the static voltage V_0 required for operating the magnetron in the manner described. (The expression you will find gives an approximation for the minimal value required for the magnetron operation; the optimal voltage is somewhat higher.)	1.1pt

Part B: The interaction of microwave radiation with water molecules (3.4 points)

This part deals with the usage of microwave radiation (radiated by the magnetron antenna into the food chamber) for cooking, that is, heating up a lossy dielectric material such as water, either pure or salty (which is our model for, say, soup).

An electric dipole is a configuration of two equal and opposite electric charges q and $-q$ a small distance d apart. The electric dipole vector points from the negative to the positive charge, and its magnitude is $p = qd$.

A time-dependent electric field $\vec{E}(t) = E(t)\hat{x}$ is applied on a single dipole of moment $\vec{p}(t)$ with constant magnitude $p_0 = |\vec{p}(t)|$. The angle between the dipole and the electric field is $\theta(t)$.

B.1	Write expressions for both the magnitude of the torque $\tau(t)$ applied by the electric field on the dipole and the power $H_i(t)$ delivered by the field to the dipole, in terms of p_0, $E(t)$, $\theta(t)$ and their derivatives.	0.5pt

Water molecules are polar, hence can be treated as electric dipoles. Due to the strong hydrogen bonds between water molecules in liquid water, one cannot treat them as independent dipoles. Rather, one should refer to the polarization vector $\vec{P}(t)$, which is the dipole moment density (average dipole moment per unit volume of an ensemble of water molecules). The polarization $\vec{P}(t)$ is parallel to the local applied alternating electric field (of the microwave radiation), $\vec{E}(t)$, and oscillates in time with an amplitude that is proportional to the amplitude of the local alternating electric field, but with a phase lag δ.

The local alternating electric field at a given location inside the water is $\vec{E}(t) = E_0 \sin(\omega t)\hat{x}$, where $\omega = 2\pi f$, giving rise to polarization $\vec{P}(t) = \beta\varepsilon_0 E_0 \sin(\omega t - \delta)\hat{x}$, where the dimensionless constant β is a property of water.

B.2 Find an expression for the time-averaged power $\langle H(t)\rangle$ per unit volume absorbed by the water. (0.5pt)

The time-average for a time dependent periodic variable $f(t)$ over its period T is defined as:

$$\langle f(t)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t. \tag{1}$$

Let us now consider the propagation of the radiation through the water. The relative dielectric constant of water (at the electromagnetic field frequency) is ε_r, and the corresponding index of refraction of water is $n = \sqrt{\varepsilon_r}$. The momentary energy density of the electric field is given by $\frac{1}{2}\varepsilon_r\varepsilon_0 E^2$. The time-averaged energy density of the electric and magnetic fields are equal.

B.3	Let us denote the time-averaged radiation energy flux density by $I(z)$ (average radiation power flow per unit area). Here z is the depth of penetration into the water, and the radiation propagates in the z direction. Find an expression for the dependence of the flux density $I(z)$ on z. The flux density at the water surface, $I(0)$, may appear in your result.	1.1pt

The phase lag δ is the result of the interaction between the water molecules. It depends on the dimensionless dielectric loss coefficient ε_ℓ and the relative dielectric constant ε_r (both of which depend on the radiation angular frequency ω and the temperature) via the relation $\tan\delta = \varepsilon_\ell/\varepsilon_r$. When δ is small enough, the electric field at penetration depth z into the water is given by:

$$\vec{E}(z,t) = \vec{E}_0 e^{-\frac{1}{2}nk_0 z\tan\delta}\sin\left(nk_0 z - \omega t\right) \tag{2}$$

where $k_0 = \omega/c$ and $c = 3.0\cdot 10^8\,\frac{m}{s}$ is the speed of light in vacuum.

B.4	Employ the approximation $\tan\delta \approx \sin\delta$ and find an expression for the coefficient β defined in Task B.2 in terms of the other parameters.	0.6pt

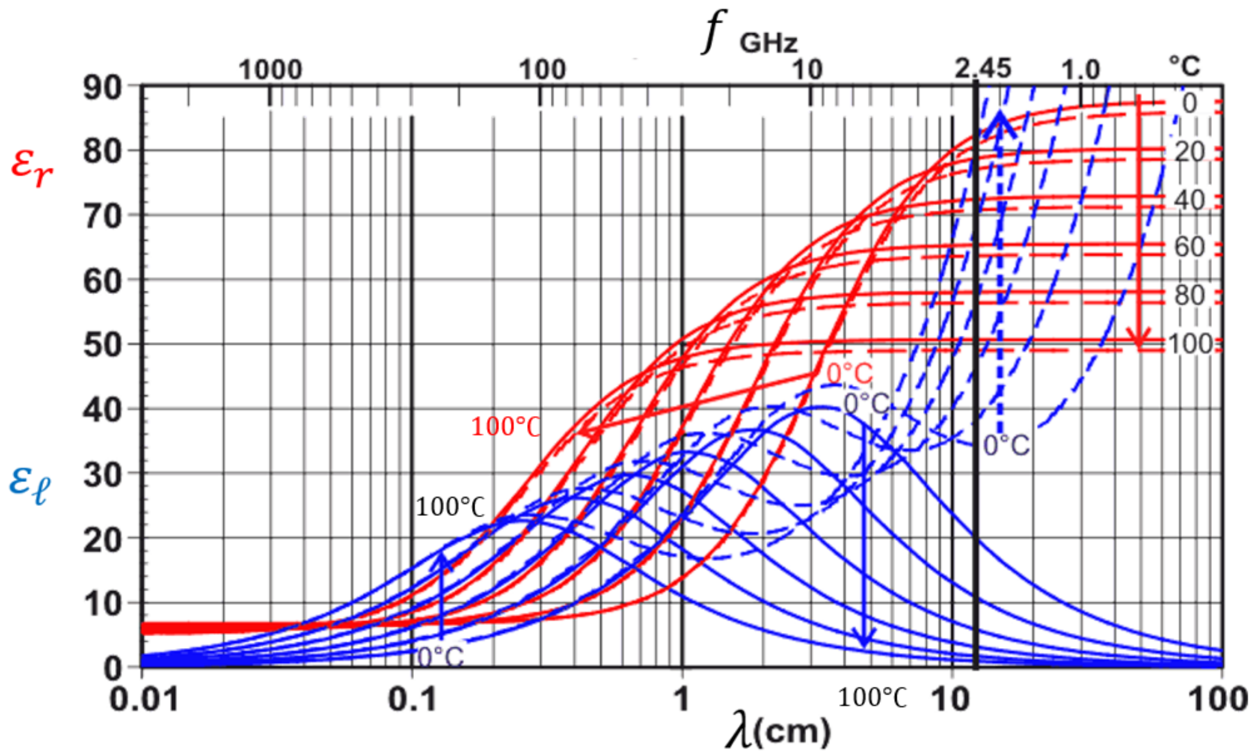

Figure 5. The arrows indicate the variation with temperature across the curves from $0^\circ\mathrm{C}$ to $100^\circ\mathrm{C}$.

Figure 5 depicts ε_ℓ (blue) and ε_r (red) for both pure water (solid lines) and a dilute solution of salt in water (dashed lines) as functions of wavelength or frequency, at several different temperatures. The angular frequency $\omega = 2\pi \cdot 2.45 \cdot 10^9\,\mathrm{s}^{-1}$ is indicated by a bold vertical line. Below we will consider microwave radiation at this frequency only.

B.5 Use Figure 5 to address the following questions: **0.7pt**

1. For water at $20^\circ\mathrm{C}$, find the penetration depth $z_{1/2}$ at which the power per unit volume is reduced to half of its value at $z = 0$.
2. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into water increases, decreases or remains the same with temperature.
3. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into soup (dilute salt solution) increases, decreases or remains the same with temperature.

Thermoacoustic Engine

A thermoacoustic engine is a device that converts heat into acoustic power, or sound waves - a form of mechanical work. Like many other heat machines, it can be operated in reverse to become a refrigerator, using sound to pump heat from a cold to a hot reservoir. The high operating frequencies reduce heat conduction and eliminate the need for any working chamber confinement. Unlike many other engine types, the thermoacoustic engine has no moving parts except the working fluid itself.

The efficiencies of thermoacoustic machines are typically lower than other engine types, but they have advantages in set up and maintenance costs. This creates opportunities for renewable energy applications, such as solar-thermal power plants and utilization of waste heat. Our analysis will focus on the creation of acoustic energy within the system, ignoring the extraction or conversion for powering external devices.

Part A: Sound wave in a closed tube (3.7 points)

Consider a thermally insulating tube of length L and cross-sectional area S, whose axis lies along the x direction. The two ends of the tube are located at $x = 0$ and $x = L$. The tube is filled with an ideal gas and is sealed on both ends. At equilibrium, the gas has temperature T_0, pressure p_0 and mass density ρ_0. Assume that viscosity can be ignored and that the gas motion is only in the x direction. The gas properties are uniform in the perpendicular y and z directions.

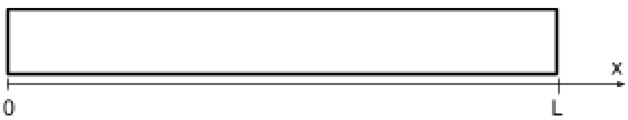

Figure 1

A.1	When a standing sound wave forms, the gas elements oscillate in the x direction with angular frequency ω. The amplitude of the oscillations depends on each element's equilibrium position x along the tube. The longitudinal displacement of each gas element from its equilibrium position x is given by $$u(x,t) = a\sin(kx)\cos(\omega t) = u_1(x)\cos(\omega t) \qquad (1)$$ (please note the u here describes the displacement of a gas element) where $a \ll L$ is a positive constant, $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength. What is the maximum possible wavelength $\lambda_{\max}$ in this system?	0.3pt

We will assume throughout the question an oscillation mode of $\lambda = \lambda_{\max}$.

Now, consider a narrow parcel of gas, located at rest between x and $x + \Delta x$ ($\Delta x \ll L$). As a result of the displacement wave of Task A.1, the parcel oscillates along the x axis and undergoes a change in volume and other thermodynamic properties.

Throughout the following tasks assume all these changes to the thermodynamic properties to be small compared to the unperturbed values.

A.2	The parcel volume $V(x,t)$ oscillates around the equilibrium value of $V_0 = S\Delta x$ and has the form $$V(x,t) = V_0 + V_1(x)\cos(\omega t). \quad (2)$$ Obtain an expression for $V_1(x)$ in terms of V_0, a, k and x.	0.5pt

A.3	Assume that the total pressure of the gas, as a result of the sound wave, takes the approximate form $$p(x,t) = p_0 - p_1(x)\cos(\omega t). \quad (3)$$ Considering the forces acting on the parcel of gas, compute the amplitude $p_1(x)$ of the pressure oscillation to leading order, in terms of the position x, the equilibrium density ρ_0, the displacement amplitude a and the wave parameters k and ω.	0.7pt

At acoustic frequencies, the thermal conductivity of the gas can be neglected. We will treat the expansion and contraction of gas parcels as purely adiabatic, satisfying the relation $pV^\gamma = \mathrm{const}$, where γ is the adiabatic constant.

A.4	Use the relation above and the results of the previous tasks to obtain an expression for the speed of sound waves $c = \omega/k$ in the tube, to first order. Express your answer in terms of p_0, ρ_0 and the adiabatic constant γ.	0.3pt

A.5	The change in the gas temperature due to the adiabatic expansion and contraction, as a result of the sound wave, takes the form: $$T(x,t) = T_0 - T_1(x)\cos(\omega t). \quad (4)$$ Compute the amplitude $T_1(x)$ of the temperature oscillations in terms of T_0, γ, a, k and x.	0.7pt

A.6	For the purpose of this task only, we assume a weak thermal interaction between the tube and the gas. As a result, the standing sound wave remains almost unchanged, but the gas can exchange a small amount of heat with the tube. The heating due to viscosity can be neglected. For each of the points in Figure 2 (A, C at the edges of the tube, B at the center) state whether the temperature of the tube at that point will increase, decrease or remain the same over a long time.	1.2pt

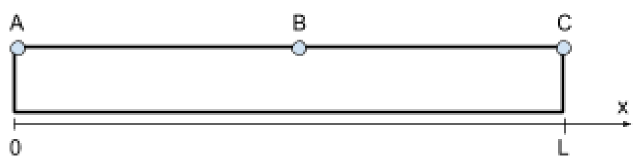

Figure 2

Part B: Sound wave amplification induced by external thermal contact (6.3 points)

A stack of thin well-spaced solid plates is placed inside the tube. The plates of the stack are aligned in parallel to the tube axis, so as not to obstruct the flow of gas along the tube. The center of the stack is positioned at $x_0 = L/4$, and spans a width of $\ell \ll L$ along the tube axis, filling its entire cross section. The right and left edges of the stack are held at temperature difference τ. The left edge of the stack, at $x_H = x_0 - \ell/2$, is held by an external thermal reservoir at temperature $T_H = T_0 + \tau/2$, and at the same time, its right edge, at $x_C = x_0 + \ell/2$, is held at a temperature $T_C = T_0 - \tau/2$.

The plate stack allows a slight longitudinal heat flow to maintain a constant temperature gradient between its edges, such that $T_{\text{plate}}(x) = T_0 - \frac{x-x_0}{\ell}\tau$.

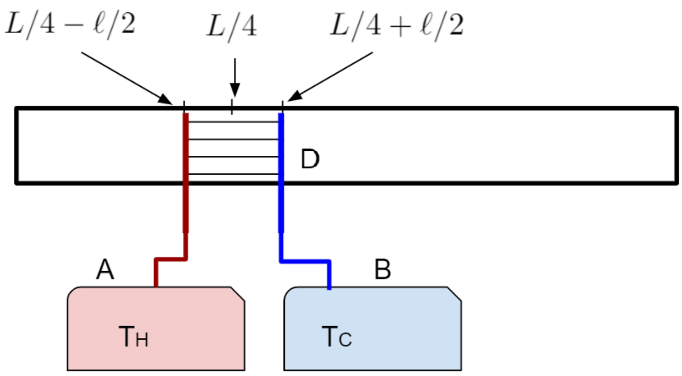

Figure 3. A sketch of the system. (A) and (B) denote the hot and cold heat reservoirs respectively. (D) denotes the stack.

To analyze the effect of the thermal contact between the plate stack and the gas on the sound waves in the tube, make the following assumptions:

- As in the previous part, all changes to the thermodynamic properties are small compared to the unperturbed values.
- The system operates in the fundamental standing-wave mode of the longest possible wavelength. It is only slightly modified by the presence of the plate stack.
- The stack is much shorter than the wavelength $\ell \ll \lambda_{\text{max}}$, and can be positioned far enough from both displacement and pressure nodes, so that the displacement $u(x,t) \approx u(x_0,t)$ and the pressure $p(x,t) \approx p(x_0,t)$ may be considered uniform over the entire length of the stack.
- We may neglect any edge effects, caused by the parcels moving in and out of the stack.
- The temperature difference between the ends of the plate stack, i.e. between the hot and the cold reservoirs, is small compared to the absolute temperature: $\tau \ll T_0$.
- Heat conduction through the stack, through the gas, and along the tube are all negligible. The only significant sources of heat transfer are convection due to the motion of the gas and conduction between the gas and the stack.

B.1	Consider a specific parcel of gas in the region of the stack, originally at $x_0 = L/4$. As the parcel moves within the stack, the local temperature of the nearby part of the stack changes as follows: $$T_{\text{env}}(t) = T_0 - T_{\text{st}} \cos(\omega t). \quad (5)$$ Express T_{st} in terms of a, τ and ℓ.	0.4pt

B.2	Above which critical temperature difference τ_{cr} will the gas be conveying heat from the hot reservoir to the cold one? Express τ_{cr} in terms of T_0, γ, k and ℓ.	1.0pt

B.3	Obtain the general approximate expression for the heat flow $\frac{dQ}{dt}$ into a small parcel of gas as a linear function of its volume and pressure change rates. Express your answer in terms of the rate of volume change $\frac{dV}{dt}$, the rate of pressure change $\frac{dp}{dt}$, the unperturbed equilibrium values of parcel pressure and volume p_0 ,V_0 and the adiabatic index γ. (You may use the expression for the molar heat capacity at constant volume $c_v = \frac{R}{\gamma - 1}$, where R is the gas constant.)	0.8pt

The limited heat flow rate between the parcel and the stack causes a phase difference between the pressure and volume oscillations of the parcel. We will see how this generates work.

Let the heat flux into the parcel from the stack be proportional to the temperature difference between the parcel and the neighboring element of the stack, given approximately by $\frac{dQ}{dt} = -\beta V_0 (T_{\text{st}} - T_1) \cos(\omega t)$. Here T_1 and T_{st} are the temperature oscillation amplitudes of the gas parcel and the neighbouring stack from Tasks A.5 and B.1, respectively, and $\beta > 0$ is a constant. Assume that at the machine's operating frequencies, the change in gas temperature as a result of this heat flow is insignificant compared to both T_1 and T_{st}.

B.4	In order to calculate work, we will consider a change to the volume of the moving parcel as a result of the thermal contact with the stack. Let us write the pressure and the volume of the parcel under the stack's influence in the form: $$\begin{aligned} p &= p_0 + p_a \sin(\omega t) - p_b \cos(\omega t), \\ V &= V_0 + V_a \sin(\omega t) + V_b \cos(\omega t). \end{aligned} \quad (6)$$ Given p_a and p_b, find the coefficients $V_a$and V_b. Express your answer in terms of p_a, p_b, p_0, V_0, γ, τ, τ_{cr}, β, ω, a and ℓ.	1.9pt

B.5	Obtain an approximate expression for the acoustic work per unit volume w produced by the gas parcel over one cycle. Integrate over the volume of the stack to obtain the total work W_{tot} generated by the gas over one cycle. Express W_{tot} in terms of γ, τ, τ_{cr}, β, ω, a, k and S.	0.8pt

B.6	Obtain an approximate expression for the heat Q_{tot} transported from the left side of the plane $x = x_0$ to the right, over a cycle. Express your answer in terms of τ, τ_{cr}, β, ω, a, S, ℓ. (Hint: you may use the formula $j = Q\frac{du}{dt}$ for the heat current due convection.)	0.8pt

B.7	Find the efficiency η of the thermoacoustic engine. The efficiency is defined as the ratio of the generated acoustic work to the heat drawn from the hot reservoir. Express your answer in terms of the temperature difference τ between the hot and the cold reservoir, the critical temperature difference τ_{cr} and the Carnot efficiency $\eta_c = 1 - T_C/T_H$.	0.6pt