# Zero-length springs and slinky coils – Solution

## Part A: Statics

A.1 The force $F$ causes the spring to change its length from $L_0$ to $L$. Since equal parts of the spring are extended to equal lengths, we get: $\frac{\Delta y}{\Delta l} = \frac{L}{L_0} \to \Delta y = \frac{L}{L_0}\Delta l$.

Since $L = \max\left\{\frac{F}{k}, L_0\right\}$, we get $\Delta y = \max\{\frac{F}{kL_0}\Delta l, \Delta l\}$. From this result we see that any piece of length $\Delta l$ the spring behaves as a ZLS with spring constant $k^* = k\frac{L_0}{\Delta l}$.

A.2 Let us compute the work of the force. From Task A.1: $dW = F(x)dx = \frac{kL_0}{\Delta l}xdx$.

Hence, $\Delta W = \int_{\Delta l}^{\Delta y}\frac{kL_0}{\Delta l}xdx = \frac{kL_0}{\Delta l}\frac{x^2}{2}\Big|_{\Delta l}^{\Delta y} = \frac{kL_0}{2\Delta l}(\Delta y^2 - \Delta l^2)$ .

A.3. At every point along the statically hanging spring the weight of the mass below is balanced by the tension from above. This implies that at the bottom of the spring there is a section of length $l_0$ whose turns are still touching each other, as their weight is insufficient to exceed the threshold force $kL_0$ to pull them apart. The length $l_0$ can be derived from the equation: $\frac{l_0}{L_0}Mg = kL_0$, hence $l_0 = \frac{kL_0^2}{Mg} = \alpha L_0$.

For $l > l_0$, a segment of the unstretched spring between $l$ and $l + \mathrm{d}l$ feels a weight of $\frac{l}{L_0}Mg$ from beneath, which causes its length to stretch from $\mathrm{d}l$ to $dy = \frac{F}{kL_0}dl = \frac{l}{L_0}Mg\frac{dl}{kL_0} = \frac{Mg}{kL_0^2}ldl = \frac{l}{l_0}dl$ .

Integration of the last expression over the stretched region, up to the point $L_0$, gives its height when the spring is stretched

$$H = l_0 + \int_{l_0}^{L_0}\frac{l}{l_0}dl = l_0 + \frac{l^2}{2l_0}\Big|_{l_0}^{L_0} = l_0 + \frac{1}{2l_0}(L_0^2 - l_0^2) = \frac{L_0^2}{2l_0} + \frac{l_0}{2} = \frac{L_0}{2}\left(\alpha + \frac{1}{\alpha}\right)$$

## Part B: Dynamics

B.1. From Task A.3 we have $H(l) = \frac{l^2}{2l_0} + \frac{l_0}{2}$. We now calculate the position of the center of mass of the suspended spring. The contribution of the unstretched section of height $l_0$ at the bottom, having a mass of $\frac{l_0}{L_0}M = \alpha M$, is $\alpha M \frac{l_0}{2}$. The position of the center of mass is obtained by summing the contributions of its elements:

$$H_{cm} = \frac{1}{M}\left[\frac{l_0}{2}\alpha M + \int_{l_0}^{L_0} H(l)dm\right] = \frac{1}{M}\left[\frac{\alpha L_0}{2}\alpha M + \int_{l_0}^{L_0}\left(\frac{l^2}{2l_0} + \frac{l_0}{2}\right)\frac{Mdl}{L_0}\right]$$
$$= \frac{\alpha^2 L_0}{2} + \frac{1}{L_0}\left[\frac{l^3}{6l_0} + \frac{l_0}{2}l\right]_{l_0}^{L_0} = \frac{\alpha^2 L_0}{2} + \frac{1}{L_0}\left[\frac{L_0^3 - l_0^3}{6l_0} + \frac{l_0}{2}(L_0 - l_0)\right]$$

Where we have used $dm = \frac{dl}{L_0}M$. Substituting $l_0 = \alpha L_0$ yields

$$H_{cm} = L_0\left[\frac{1}{6\alpha} - \frac{\alpha^2}{6} + \frac{\alpha}{2}\right]$$

When the spring is contracted to its free length $L_0$, its center of mass is located at $\frac{L_0}{2}$. From the falling of the center of mass at acceleration $g$ we get:

$$\frac{g}{2}t_c^2 = H_{cm} - \frac{L_0}{2} = L_0\left[\frac{1}{6\alpha} - \frac{\alpha^2}{6} + \frac{\alpha}{2} - \frac{1}{2}\right] = \frac{L_0}{6\alpha}(1-\alpha)^3$$

Hence, $t_c = \sqrt{\frac{L_0}{3g\alpha}(1-\alpha)^3}$.

For $k = 1.02$ N/m, $L_0 = 0.055$ m, $M = 0.201$ kg, and $g = 9.80$ m/s$^2$, we have $\alpha = 0.0285$, and $t_c = 0.245$ s.

B.2. The moving top section of the spring is pulled down by its own weight, $m_{top}g = Mg\frac{(L_0-l)}{L_0}$ and also by the tension in the spring below, which is equal to the weight $Mgl/L_0$ of the stationary section of the spring. Thus, the moving top section experiences a constant force $F = Mg$ throughout its whole fall. Another way to see that, is that a total force of $Mg$ is exerted on the spring, but only the moving part experiences it. Let's calculate the position of the center of mass at equilibrium of the upper part, i.e., all points with $l' > l$ for some $l > l_0$. From part A, the position of a small portion $\Delta l'$ with coordinate $l'$ is: $H(l') = \frac{l'^2}{2l_0} + \frac{l_0}{2}$ and the center of mass of this part is:

$$H_{cm-upper-i} = \frac{L_0}{M(L_0-l)}\int_l^{L_0} H(l')dm = \frac{L_0}{M(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)dm$$

$$= \frac{L_0}{M(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)\frac{Mdl'}{L_0} = \frac{1}{(L_0-l)}\int_l^{L_0}\left(\frac{l'^2}{2l_0}+\frac{l_0}{2}\right)dl'$$

$$= \frac{1}{(L_0-l)}\left[\frac{l'^3}{6l_0}+\frac{l_0 l'}{2}\right]_l^{L\_0} = \frac{L_0^2+L_0 l+l^2}{6l_0}+\frac{l_0}{2}$$

The position of the upper part of CM when it contracts to a length $L_0 - l$ is $H_{cm-upper-f} = \frac{l^2}{2l_0} + \frac{l_0}{2} + \frac{1}{2}(L_0 - l)$. The change in the CM during the contraction process is: $\Delta H_{cm-upper} = H_{cm-upper-i} - H_{cm-upper-f} = \frac{L_0^2+L_0 l-2l^2}{6l_0} - \frac{1}{2}(L_0 - l) = \frac{(L_0-l)(L_0+2l)}{6l_0} - \frac{1}{2}(L_0 - l)$.

The acceleration of the CM of the upper part is $a_{CM} = \frac{FL_0}{M(L_0-l)} = \frac{gL_0}{L_0-l}$.
From the work energy theorem we get the equation $v_{upper-f}^2 = 2a_{CM}\Delta H_{cm-upper}$, hence

$$v_{upper-f}^2 = 2\frac{gL_0}{L_0-l}\left[\frac{(L_0-l)(L_0+2l)}{6\alpha L_0} - \frac{1}{2}(L_0-l)\right] = 2g\left[\frac{L_0+2l}{6\alpha} - \frac{1}{2}L_0\right]$$

$$= \frac{2g}{3\alpha}l + \left(\frac{1}{3\alpha}-1\right)gL_0$$

Therefore, $A = \frac{2g}{3\alpha}$ and $B = \left(\frac{1}{3\alpha} - 1\right)gL_0$.

Note that for $l = L_0$, we have $v_{upper-f}^2 = L_0 g\frac{1-\alpha}{\alpha}$ and for $l = l_0 = \alpha L_0$, we get $v_{upper-f}^2 = L_0 g\frac{1-\alpha}{3\alpha}$, hence, the moment we release the spring its velocity is finite (not zero, the meaning is that it accumulate this velocity in time that is much shorter than the contracting time $t_c$) and it decreases to $\frac{1}{\sqrt{3}}$ of the initial value when $l = l_0$.

B.3. Note that even though the center of mass of the spring accelerates downwards constantly, the moving top section actually decelerates, while the position of the center of mass moves down the spring. The speed of the top section $v(l)$, calculated in Task B2, decreases and approaches the value $\sqrt{A\alpha L_0 + B}$ immediately before it attaches to the bottom section of height $l_0 = \alpha L_0$, which was unstretched and at rest. Once the moving top section attaches to the resting bottom section, its momentum is shared between both sections, so the speed further decreases just before the whole spring starts accelerating downwards as a single mass. Thus, the minimum speed is that of the whole spring immediately after its full collapse. From momentum conservation, we have

$$Mv_{min} = m_{top}v(l_0) = M\left(1 - \frac{l_0}{L_0}\right)\sqrt{A\alpha L_0 + B}$$

$$v_{min} = (1-\alpha)\sqrt{A\alpha L_0 + B}$$

## Part C: Energetics

C.1. From the moment the spring is released, the acceleration of its center of mass is governed by the external force $Mg$ and therefore the gravitational potential energy of the spring is fully converted into the kinetic energy of the center of mass of the spring, which just before hitting the ground is equal to the kinetic energy of the spring.

All that is left is the elastic energy stored in the spring, which is converted into heat, sound, etc. To calculate it, we consider the elastic energy stored in a segment $dh$ of the stretched spring, which when unstretched lies between $l$ and $l + \mathrm{d}l$, using the result of Task A.2, $\Delta W = \frac{kL_0}{2\Delta l}(\Delta l_2^2 - \Delta l^2)$, by choosing $\Delta l = dl$ and $\Delta l_2 = dy$, and using $dy = \frac{l}{l_0}dl$ (which was obtained in Task A.3), we get:

$dW = \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl$. Integrating from $l_0$ to $L_0$ we find

$$W = \int_{l_0}^{L_0} \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl = \frac{kL_0}{2}\left[\frac{l^3}{3l_0^2} - l\right]_{l_0}^{L_0} = \frac{kL_0}{2}\left(\frac{L_0^3 - l_0^3}{3l_0^2} - (L_0 - l_0)\right)$$

$$= \frac{kL_0^2}{2}\left(\frac{1-\alpha^3}{3\alpha^2} - (1-\alpha)\right) = \frac{kL_0^2}{6\alpha^2}(1-\alpha)^2(2\alpha+1)$$

$$= MgL_0\frac{(1-\alpha)^2(2\alpha+1)}{6\alpha}$$

# The Physics of a Microwave Oven – Solution

## Part A: The structure and operation of a magnetron

A.1. The frequency of an LC circuit is $f = \omega/2\pi = 1/(2\pi\sqrt{LC})$. If the total electric current flowing along the boundary of the cavity is $I$, it generates a magnetic field whose magnitude (by the assumptions of the question) is $0.6\mu_0 I/h$, and a total magnetic flux equal to $\pi R^2 \times 0.6\mu_0 I/h$, hence the inductance of the resonator is $L = 0.6\pi\mu_0 R^2/h$. Approximating the capacitor as a plate capacitor, its capacitance is $C = \varepsilon_0 lh/d$. Putting everything together, we find

$$f_{est} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} = \frac{1}{2\pi}\sqrt{\frac{h}{0.6\pi R^2\mu_0}\frac{d}{\varepsilon_0 lh}} = \frac{1}{2\pi}\frac{c}{R}\sqrt{\frac{d}{0.6\pi l}} = \frac{1}{2\pi}\frac{3\cdot 10^8}{7\cdot 10^{-3}}\sqrt{\frac{1}{3.6\pi}} = 2.0\cdot 10^9 \text{ Hz}$$

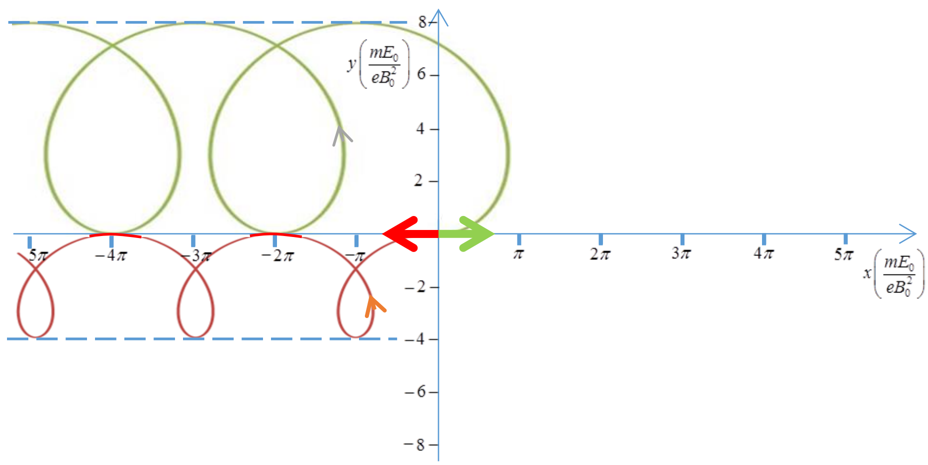
A.2. Denoting the electron velocity by $\vec{u}(t)$, in this case the total force applied on it is

$$\vec{F} = -e(-E_0\hat{y} + \vec{u}(t) \times B_0\hat{z}).$$

Let us write $\vec{u}(t) = \vec{u}_D + \vec{u}'(t)$, with $\vec{u}_D = (-E_0/B_0)\hat{x}$ being the drift velocity of a charged particle in the crossed electric and magnetic fields (the velocity at which the electric and magnetic forces cancel each other exactly). Then $\vec{F} = -e\vec{u}'(t) \times B_0\hat{z}$. Thus, in a frame moving at the drift velocity $\vec{u}_D$, the electron trajectory is a circle with constant-magnitude velocity $u' = |\vec{u}'(t)|$, and radius $r = mu'/eB_0$. In the lab frame this circular motion is superimposed upon the drift at the constant velocity $\vec{u}_D$. Hence:

1. For $\vec{u}(0) = (3E_0/B_0)\hat{x}$ we find $u' = 4E_0/B_0$ and $r = 4mE_0/eB_0^2$.
2. For $\vec{u}(0) = -(3E_0/B_0)\hat{x}$ we find $u' = 2E_0/B_0$ and $r = 2mE_0/eB_0^2$.

This information, together with the independence of the period of the circular motion on $u'$ allows us to plot the electron trajectory in both cases (green and red, for cases 1 and 2, respectively):



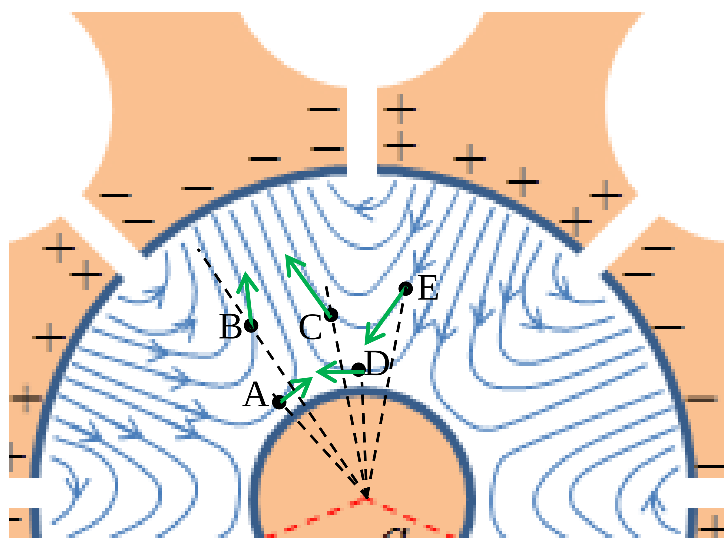
A.3. The velocity of the electron in a frame of reference where the motion is approximately circular is $u'$. From A.2 we get that $u_D + u' = v_{\max}$ and $u_D - u' = v_{\min}$, hence $u' = (v_{\max} - v_{\min})/2 < v_{\max}$.

The radius of the circular motion of the electron in this frame is $r = mu'/eB_0 < mv_{\max}/eB_0$. The maximal velocity is that corresponding to a kinetic energy, $K_{\max} = mv_{\max}^2/2$, of 800 eV.

Substituting we find $r < \frac{m}{eB}\sqrt{\frac{2eV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{e}} = \frac{1}{0.3}\sqrt{\frac{2\cdot 9.1\cdot 10^{-31}\cdot 800}{1.6\cdot 10^{-19}}} = 3.18\cdot 10^{-4}\ \mathrm{m} \approx 0.3\mathrm{mm}$.

Since this maximal radius is much smaller than the distance between the anode and the cathode, we may ignore the circular component of the electronic motion, and approximate it as pure drift.

A.4. As just explained, we may approximate the electron motion as pure drift. In task A.2 we have found that the direction of the drift velocity $\vec{u}_D$ is in the direction of the vector $\vec{E} \times \vec{B}$. Since we are interested in radial component of the drift velocity, the only contribution is from the azimuthal component of the electric field. The static electric field has no azimuthal component, hence the drift in the radial direction results solely from the azimuthal component of the alternating electric field. What we have to check is if the azimuthal component points clockwise or counterclockwise. From the direction of the field lines it is easy to see (attached figure) that in points A and B the azimuthal component pointing clockwise therefore the electrons there drift towards the cathode, while for points C, D and E the azimuthal component points counterclockwise and the electrons there drift toward the anode.



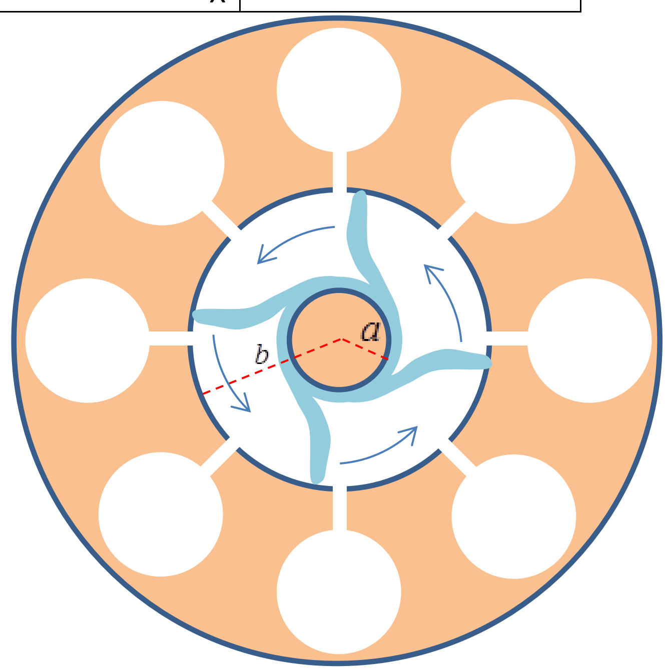

| Point | toward the anode | toward the cathode | perpendicular to the radius |
|---|---|---|---|
| A | | X | |
| B | | X | |

| | | | |
|---|---|---|---|
| C | X | | |
| D | X | | |
| E | X | | |

A.5. In this task we need to consider the azimuthal component of the drift velocity, which results from the radial component of the electric field. Since all points are at the same distance from the anode, all electrons experience the same static electric field. Hence only the radial component of the alternating field determines whether the angle between the electrons' position vectors would increase or decrease: If the radial component of the alternating field points inwards (towards the cathode), the azimuthal drift velocity will be positive (counterclockwise) and vice versa. Hence the electrons at A, B and C drift closer to each other in terms of angles, while those at D, E and F drift away from each other.

| points | angle decreases | angle increases | indeterminate |
|---|---|---|---|
| AB | X | | |
| BC | X | | |
| CA | X | | |
| DE | | X | |
| EF | | X | |
| DF | | X | |

A.6. Spokes will be created only in the regions where focusing occurs. By the result of the previous task, there are four spokes, as indicated in the attached Figure.




The electron drift sets the spokes in a counterclockwise rotation. The frequency of the alternating field is $f = 2.45$ GHz. By the time the alternating field flipped its sign (half a period), each spoke moves to the next cavity, corresponding to an angle of $\pi/4$. Therefore, the angular velocity of each spoke is

$\omega = \frac{\pi}{4}/\frac{T}{2} = \frac{\pi}{2} f = 3.85 \cdot 10^9 rad/s$. Each spoke performs a full rotation around the magnetron after four periods of the alternating field.

A.7. The magnitude of the electric field in the region considered, $r = (b+a)/2$, is the magnitude of the static field, that is, $E = V_0/(b-a)$, giving rise to an azimuthal drift velocity of magnitude $u_D = E/B_0 = V_0/[B_0(b-a)]$. Equating $u_D/r$ with the angular velocity found in the previous task we find $V_0 = \pi fB_0(b^2 - a^2)/4$

## Part B: The interaction of microwave radiation with water molecules

B.1. The torque at time $t$ is given by $\tau(t) = -qd\sin[\theta(t)]E(t) = -p_0\sin[\theta(t)]E(t)$, hence the instantaneous power delivered to the dipole by the electric field is

$$H_i(t) = \tau(t)\dot{\theta}(t) = -p_0E(t)\sin\theta(t)\dot{\theta}(t) = E(t)\frac{d}{dt}\left(p_0\cos\theta(t)\right) = E(t)\frac{dp_x(t)}{dt}$$

B.2. Since the average dipole density (hence the average of each molecular dipole) is parallel to the field, the absorbed power density is (angular brackets, $\langle\cdots\rangle$, denote average over time)

$$\left\langle H(t)\right\rangle = \left\langle E_0\sin(\omega_f t)\frac{dP_x}{dt}\right\rangle = \left\langle E_0\sin(\omega_f t)\frac{d}{dt}\left(\beta\varepsilon_0 E_0\sin(\omega_f t-\delta)\right)\right\rangle =$$

$$E_0^2\beta\varepsilon_0\omega_f\left\langle\sin(\omega_f t)\cos(\omega_f t-\delta)\right\rangle = 0.5E_0^2\beta\varepsilon_0\omega_f\left\langle\sin\delta+\sin(2\omega_f t-\delta)\right\rangle = 0.5E_0^2\beta\varepsilon_0\omega_f\sin\delta$$

B.3. The energy density of the electromagnetic field at penetration depth $z$, which is twice the electric energy density, is $2\times\varepsilon_r\varepsilon_0\langle E^2(z,t)\rangle/2 = \varepsilon_r\varepsilon_0E_0^2(z)\langle\sin^2(\omega t)\rangle = \varepsilon_r\varepsilon_0E_0^2(z)/2$. Therefore, the time-averaged flux density at depth $z$ is:

$$I(z) = \frac{1}{2}\varepsilon_r\varepsilon_0E_0^2(z)\times\frac{c}{n} = \frac{1}{2}\sqrt{\varepsilon_r}\varepsilon_0cE_0^2(z),$$

where $c$ is the speed of light in vacuum. $I$ decreases with $z$ due to the absorbed power calculated in the previous task we find

$$\frac{dI(z)}{dz} = -\frac{1}{2}\beta\varepsilon_0\omega E_0^2(z)\sin\delta = -\frac{\beta\omega\sin\delta}{c\sqrt{\varepsilon_r}}I(z),$$

hence $I(z) = I(0)\exp[-z\beta\omega\sin\delta/(c\sqrt{\varepsilon_r})]$.

B.4. Similarly to the previous task, the energy flux corresponding to the given field is

$$I(z) = \sqrt{\varepsilon_r}\varepsilon_0c\langle E^2(z,t)\rangle = \frac{1}{2}\sqrt{\varepsilon_r}\varepsilon_0cE_0^2e^{-z\omega\sqrt{\varepsilon_r}\tan\delta/c}.$$

Equating the argument of the exponent in the last expression with the result of the previous task, and using the given approximation $\tan\delta\approx\sin\delta$ leads to $\beta = \varepsilon_r$.

B.5.

1. Using previous results, the radiation power per unit area is reduced to half of its $z = 0$ value at $z_{1/2} = c \ln 2 / (\omega\sqrt{\varepsilon_r} \tan\delta) = c\sqrt{\varepsilon_r} \ln 2 / (\omega\varepsilon_l)$. From the given graph, at the given frequency $\varepsilon_r \approx 78$ and $\varepsilon_l \approx 10$, hence $z_{1/2} \approx 12$ mm.

We have just found that the penetration depth is proportional to $\sqrt{\varepsilon_r}/\varepsilon_l$. From the given graph we thus find that:

2. Heating up pure water (continuous lines) decreases $\varepsilon_l$ much more significantly than the corresponding decrease of $\sqrt{\varepsilon_r}$ at the given frequency. Thus, the penetration depth of pure water increases with temperature, allowing deeper penetration of the microwave radiation and heating up the water inner regions.

3. On the contrary, for a soup (dilute salt solution, dashed lines) $\varepsilon_l$ at the given frequency increases with temperature while $\varepsilon_r$ decreases. Thus, the absorption rate increases with temperature, the penetration depth decreases, and less microwave radiation reaches its inner regions.

| material | $z_{1/2}$increases with temp. | $z_{1/2}$ decreases with temp. | $z_{1/2}$remains the same |
|---|---|---|---|
| water | X | | |
| soup | | X | |

# Thermoacoustic engine – Solution

## Part A: Sound wave in a closed tube

A.1. The boundary conditions are: $u(0,t) = u(L,t) = 0$. As a result, $\sin\left(\frac{2\pi}{\lambda}L\right) = 0$, so we get $\lambda_{\text{max}} = 2L$.

A.2. We get

$$V(x,t) = S \cdot \big(\Delta x + u(x+\Delta x, t) - u(x,t)\big) = S\Delta x \cdot (1+u') = V_0 + V_0 u'.$$

Thus,

$$V(x,t) = V_0 + akV_0\cos(kx)\cos(\omega t) \qquad \Rightarrow \qquad V_1(x) = akV_0\cos(kx).$$

A.3. We use Newton's Second Law $\rho_0\ddot{u} = -p'$ to deduce $p' = -\rho_0\ddot{u} = \rho_0 a\omega^2\sin(kx)\cos(\omega t)$, so that

$$p(x,t) = p_0 - a\frac{\omega^2}{k}\rho_0\cos(kx)\cos(\omega t) \qquad \Rightarrow \qquad p_1(x) = a\frac{\omega^2}{k}\rho_0\cos(kx).$$

A.4. Using $a \ll L$, we obtain $\frac{p_1(x)}{p_0} = \gamma\frac{V_1(x)}{V_0}$. As a result, $\frac{\rho_0}{p_0}\frac{\omega^2}{k} = \gamma \cdot k$, and $c = \sqrt{\frac{\gamma p_0}{\rho_0}}$.

A.5. The relative change in $T(x,t)$ is the sum of the relative changes in $V(x,t)$ and $p(x,t)$. As a result,

$$T_1(x) = \frac{T_0}{p_0}p_1(x) - \frac{T_0}{V_0}V_1(x) = (\gamma-1)\frac{T_0}{V_0}V_1(x) = ak(\gamma-1)T_0\cos(kx).$$

A.6. The movement of the gas parcels inside the tube conveys heat along its boundary. To determine the direction of the convection, we combining the result of Task A.5 and the expression (1) for $u(x,t)$. We see that when $0 < x < \frac{L}{2}$, the gas is colder when the displacement $u(x,t)$ is positive. Likewise, when $\frac{L}{2} < x < L$, the gas is colder when the displacement $u(x,t)$ is negative. Hence, heat flows into the gas near the point B, cooling it down, and out of the gas near the points A and C, heating them up.

## Part B: Sound wave amplification induced by external thermal contact

B.1. We get

$$T_{\text{env}}(t) = T_{\text{plate}}\big(x_0 + u(x_0,t)\big) = T_0 - \frac{\tau}{\ell}\cdot u(x_0,t),$$

so that:

$$T_{\text{st}} = \frac{a\tau}{\ell}\sin(kx_0) = \frac{a\tau}{\ell\sqrt{2}}.$$

B.2. The gas will convey heat from the hot reservoir to the cold one if the parcels are colder than the environment when $u(x_0, t) < 0$, and hotter when $u(x_0, t) > 0$. This occurs precisely if

$$T_{\text{st}} > T_1.$$

Plugging in the results of Tasks A.5 and B.1, we get

$$\frac{a\tau_{\text{cr}}}{\ell}\sin(kx_0) = ak(\gamma - 1)T_0\cos(kx_0) \quad \Rightarrow \quad \tau_{\text{cr}} = k\ell(\gamma - 1)T_0.$$

B.3. Using the first law of thermodynamics, we get

$$\frac{dQ}{dt} = \frac{dE}{dt} + p\frac{dV}{dt}.$$

Plugging in the relation $E = \frac{1}{\gamma-1}pV$, we see that:

$$\frac{dQ}{dt} = \frac{1}{\gamma-1}\frac{d}{dt}(pV) + p\frac{dV}{dt} = \frac{1}{\gamma-1}V\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p\frac{dV}{dt} \approx \frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt}.$$

B.4. We plug the expression for $\frac{dQ}{dt}$ into the result of Task B.3. This gives:

$$\frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt} = \beta V_0(T_{\text{st}} - T_1)\cdot\cos(\omega t).$$

We now plug in the data given in equation (6), and get (by considering terms with $\cos(\omega t)$ and $\sin(\omega t)$ separately):

$$\frac{1}{\gamma - 1}V_0 p_a\omega + \frac{\gamma}{\gamma - 1}p_0 V_a\omega = \beta V_0(T_{\text{st}} - T_1)$$

$$\frac{1}{\gamma - 1}V_0 p_b\omega - \frac{\gamma}{\gamma - 1}p_0 V_b\omega = 0$$

and thus, we can already express $V_b$ as

$$V_b = \frac{1}{\gamma}p_b\cdot\frac{V_0}{p_0}.$$

For $V_a$, we plug in the results of Tasks B.1 and B.2,

$$T_{\text{st}} - T_1 = \frac{a}{\ell\sqrt{2}}(\tau - \tau_{\text{cr}}),$$

giving:

$$V_a = \left(-\frac{1}{\gamma}p_a - \frac{\gamma-1}{\gamma}\frac{\beta}{\omega}\frac{a}{\ell\sqrt{2}}(\tau - \tau_{\mathrm{cr}})\right)\cdot\frac{V_0}{p_0}.$$

B.5. We want to integrate the mechanical work generated, $\int pdV$, and averaging the result over a long time. To do this, we substitute our expressions (6) for the perturbed $p$ and $V$. Since the average of $\cos(\omega t)\sin(\omega t)$ is 0, and that of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ is $\frac{1}{2}$, we get:

$$\frac{V_0}{S\ell}W_{\mathrm{tot}} = -\pi\cdot(p_aV_b + p_bV_a).$$

Using the result of B.4, we get

$$\frac{V_0}{S\ell}W_{\mathrm{tot}} = \frac{\pi}{\omega}\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\ell\sqrt{2}}(\tau - \tau_{\mathrm{cr}})\cdot V_0\frac{p_b}{p_0}.$$

To leading order, $p_b$ is the unperturbed wave $p_b \approx p_1(x_0) = a\frac{\omega^2}{k}\rho_0\cos(kx_0) = ak\gamma p_0\frac{1}{\sqrt{2}}$. Simplifying, we get

$$W_{\mathrm{tot}} = \frac{\pi}{\omega}S\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\sqrt{2}}(\tau - \tau_{\mathrm{cr}})\cdot\frac{p_b}{p_0} = \frac{\pi}{2\omega}(\gamma-1)\beta(\tau-\tau_{\mathrm{cr}})ka^2S.$$

B.6. We want to compute the amount of heat convection over one cycle. This means that we need to take the amount of heat moving in or out of the parcel, and weigh it by the position of the parcel at that time. Thus, the total heat conveyed by the parcel, integrated along a cycle, is:

$$Q_{\mathrm{tot}} = \frac{1}{\Delta x}\int\frac{dQ}{dt}u\cdot dt.$$

This expression can be computed to leading order using $\frac{dQ}{dt} = \beta V_0(T_{\mathrm{st}} - T_1)\cdot\cos(\omega t)$ and the unperturbed displacement $u(x_0,t) = \frac{a}{\sqrt{2}}\cos(\omega t)$. This gives

$$Q_{\mathrm{tot}} = \frac{\pi}{\omega}\beta V_0(T_{\mathrm{st}} - T_1)\frac{a}{\sqrt{2}} = \frac{\pi}{\omega}\beta V_0\cdot\frac{a}{\ell\sqrt{2}}(\tau-\tau_{\mathrm{cr}})\cdot\frac{a}{\sqrt{2}} = \frac{\pi}{2\omega}\beta(\tau-\tau_{\mathrm{cr}})\frac{a^2S}{\ell}.$$

B.7. Dividing the results of Tasks B.5 and B.6, we obtain the expression:

$$\eta = \frac{W_{\mathrm{tot}}}{Q_{\mathrm{tot}}} = (\gamma-1)k\ell = \frac{\tau_{\mathrm{cr}}}{T_0} = \frac{\tau_{\mathrm{cr}}}{\tau}\cdot\frac{\tau}{T_0} = \frac{\tau_{\mathrm{cr}}}{\tau}\cdot\eta_c.$$