Planetary Physics (10 points)

This problem consists of two independent problems related to the interior of planets. The effects of the surface curvature of the planets can be neglected. You might need to use the formula

$$(1+x)^{\varepsilon} \approx 1+\varepsilon x, \text{ when } |x| \ll 1. \tag{1}$$

Part A. Mid-ocean ridge (5.0 points)

Consider a large vessel of water that is situated in a uniform gravitational field with free-fall acceleration g. Two vertical rectangular plates parallel to each other are fitted into the vessel so that the vertical edges of the plates are in a tight gap-less contact with the vertical walls of the vessel. Length h of each plate is immersed in water (Fig. 1). The width of the plates along the y-axis is w, water density is ρ_0.

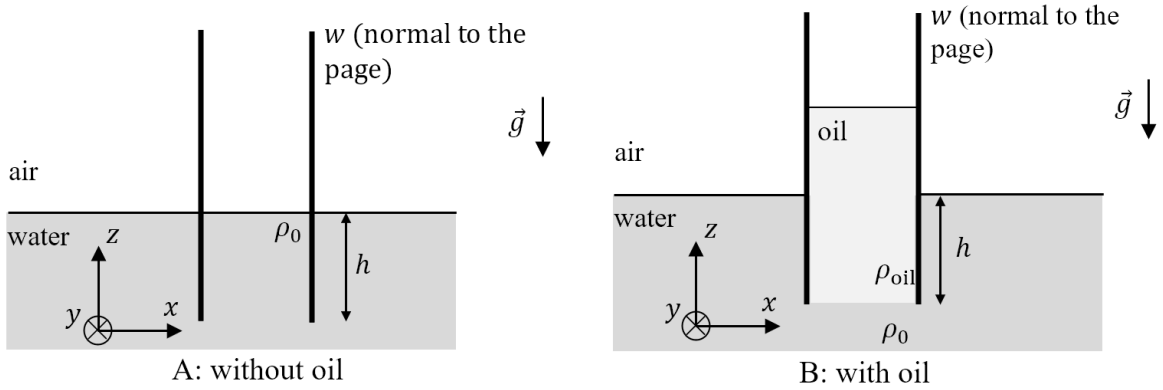

Figure 1. Parallel plates in water.

Oil of density ρ_{oil} ($\rho_{\text{oil}} < \rho_0$) is poured into the space between the plates until the lower level of the oil has reached the lower edges of the plates. Assume that plates and vessel edges are high enough for oil not to overflow them. Surface tension and mixing of fluids can be neglected.

A.1	What is the x-component of the net force F_x acting on the right plate (magnitude and direction)?	0.8pt

Fig. 2 shows a cross-section of a mid-ocean ridge. It consists of overlaying layers of mantle, crust and ocean water. The mantle is composed of rocks that we assume can flow in geological timescales and therefore, in this problem will be treated as a fluid. The thickness of the crust is much smaller than the characteristic length scale in the x-direction, hence, the crust behaves as a freely bendable plate. To high accuracy, such a ridge can be modeled as a two-dimensional system, without any variation of variables along the y-axis, which is perpendicular to the plane of Fig. 2. Assume that the ridge length L along the y-axis is much larger than any other length introduced in this problem.

At the centre of the ridge the thickness of the crust is assumed to be zero. As the horizontal distance x from the centre increases, the crust gets thicker and approaches a constant thickness D as $x \to \infty$. Correspondingly, the ocean floor subsides by a vertical height h below the top of the ridge O, which we define as the origin of our coordinate system (see Fig. 2). Water density ρ_0 and temperature T_0 can be assumed to be constant in space and time. The same can be assumed for mantle density ρ_1 and its temperature T_1. The temperature of the crust T is also constant in time but can depend on position.

It is known that, to high accuracy, the crustal material expands linearly with temperature T. Since water and mantle temperatures are assumed to be constant, it is convenient to use a rescaled version of the thermal expansion coefficient. Then $l(T) = l_1 \left[1 - k_l (T_1 - T) / (T_1 - T_0)\right]$, where l is the length of a piece of crustal material, l_1 is its length at temperature T_1, and k_l is the rescaled thermal expansion coefficient, which can be assumed to be constant.

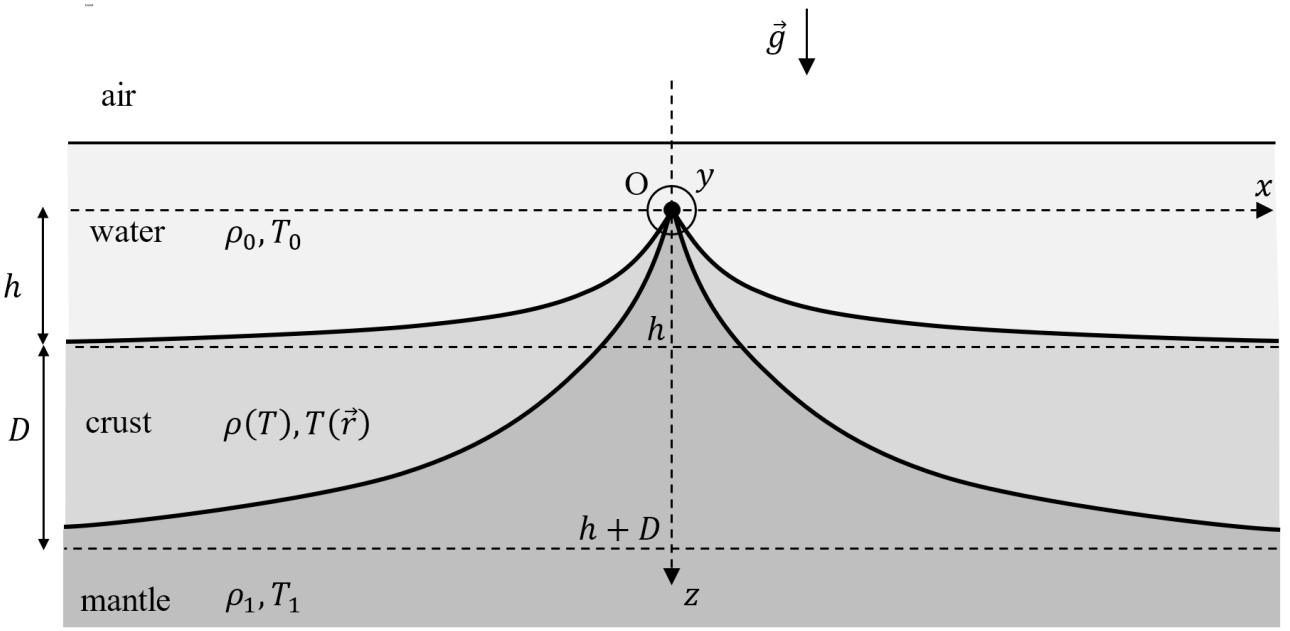

Figure 2. Mid-ocean ridge. Note that the z-axis is pointing downwards.

A.2 Assuming that the crust is isotropic, find how its density ρ depends on its temperature T. Assuming that $|k_l| \ll 1$, write your answer in the approximate form 0.6pt

$$\rho(T) \approx \rho_1 \left[1 + k \frac{T_1 - T}{T_1 - T_0}\right], \tag{2}$$

where terms of order k_l^2 and higher are neglected. Then, identify constant k.

It is known that $k > 0$. Also, thermal conductivity of the crust κ can be assumed to be constant. As a consequence, very far away from the ridge axis the temperature of the crust depends linearly with depth.

A.3 By assuming that mantle and water each behave like an incompressible fluid at hydrostatic equilibrium, express the far-distance crust thickness D in terms of h, ρ_0, ρ_1, and k. Any motion of the material can be neglected. 1.1pt

A.4 Find, to the leading order in k, the net horizontal force F acting on the right half ($x > 0$) of the crust in terms of ρ_0, ρ_1, h, L, k and g. 1.6pt

Suppose that crust is thermally isolated from the rest of the Earth. As a result of heat conduction, the temperatures of the upper and lower surfaces of the crust are going to get closer to each other until the

crust reaches thermal equilibrium. Specific heat of the crust is c and can be assumed to be constant.

A.5	By using dimensional analysis or order-of-magnitude analysis, estimate the characteristic time τ in which the difference between the upper and lower surface temperatures of the crust far away from the ridge axis is going to approach zero. You can assume that τ does not depend on the two initial surface temperatures of the crust.	0.9pt

Part B. Seismic waves in a stratified medium (5.0 points)

Suppose that a short earthquake happens at the surface of some planet. The seismic waves can be assumed to originate from a line source situated at $z = x = 0$, where x is the horizontal coordinate and z is the depth below the surface (Fig. 3). The seismic wave source can be assumed to be much longer than any other length considered in this question.

As a result of the earthquake, a uniform flux of the so-called longitudinal P waves is emitted along all the directions in the x-z plane that have positive component along the z-axis. Since the wave theory in a solid is generally complicated, in this problem we neglect all the other waves emitted by the earthquake. The crust of the planet is stratified so that the P-wave speed v depends on depth z according to $v = v_0(1+z/z_0)$, where v_0 is the speed at the surface and z_0 is a known positive constant.

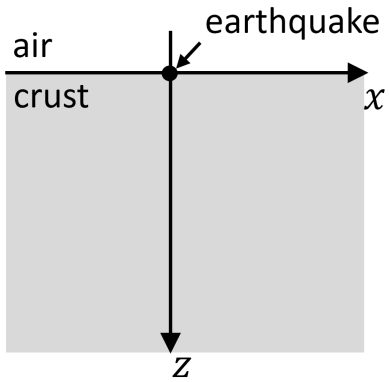

Figure 3. Coordinate system used in part B.

B.1	Consider a single ray emitted by the earthquake that makes an initial angle $0 < \theta_0 < \pi/2$ with the z-axis and travels in the x-z plane. What is the horizontal coordinate $x_1(\theta_0) \neq 0$ at which this ray can be detected at the surface of the planet? It is known that the ray path is an arc of a circle. Write your answer in the form $x_1(\theta_0) = A\cot(b\theta_0)$, where A and b are constants to be found.	1.5pt

If you were unable to find A and b, in the following questions you can use the result $x_1(\theta_0) = A\cot(b\theta_0)$ as given. Suppose that total energy per unit length of the source released as P waves into the crust during the earthquake is E. Assume that waves are completely absorbed when they reach the surface of the planet from below.

B.2	Find how the energy density per unit area $\varepsilon(x)$ absorbed by the surface depends on the distance along the surface x. Sketch the plot of $\varepsilon(x)$.	1.5pt

From now on, assume that the waves are instead fully reflected when reaching the surface. Imagine a

device positioned at $z = x = 0$ that has the same geometry as the previously considered earthquake source. The device is capable of emitting P waves in a freely chosen angular distribution. We make the device emit a signal with a narrow range of emission angles. In particular, the initial angle the signal makes with the vertical belongs to the interval $[\theta_0 - \frac{1}{2}\delta\theta_0, \theta_0 + \frac{1}{2}\delta\theta_0]$, where $0 < \theta_0 < \pi/2$, $\delta\theta_0 \ll 1$ and $\delta\theta_0 \ll \theta_0$.

B.3	At what distance x_{max} along the surface from the source is the furthest point that the signal does not reach? Write your answer in terms of θ_0, $\delta\theta_0$ and other constants given above.	2.0pt

Electrostatic lens (10 points)

Consider a uniformly charged metallic ring of radius R and total charge q. The ring is a hollow toroid of thickness $2a \ll R$. This thickness can be neglected in parts A, B, C, and E. The xy plane coincides with the plane of the ring, while the z-axis is perpendicular to it, as shown in Figure 1. In parts A and B you might need to use the formula (Taylor expansion)

$$(1+x)^{\varepsilon} \approx 1 + \varepsilon x + \frac{1}{2}\varepsilon(\varepsilon-1)x^2, \text{ when } |x| \ll 1.$$

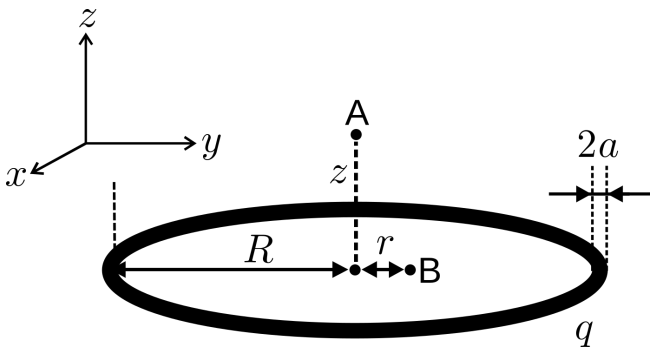

Figure 1. A charged ring of radius *R*.

Part A. Electrostatic potential on the axis of the ring (1 point)

A.1	Calculate the electrostatic potential $\Phi(z)$ along the axis of the ring at a z distance from its center (point A in Figure 1).	0.3pt
A.2	Calculate the electrostatic potential $\Phi(z)$ to the lowest non-zero power of z, assuming $z \ll R$.	0.4pt
A.3	An electron (mass m and charge $-e$) is placed at point A (Figure 1, $z \ll R$). What is the force acting on the electron? Looking at the expression of the force, determine the sign of q so that the resulting motion would correspond to oscillations. The moving electron does not influence the charge distribution on the ring.	0.2pt
A.4	What is the angular frequency ω of such harmonic oscillations?	0.1pt

Part B. Electrostatic potential in the plane of the ring (1.7 points)

In this part of the problem you will have to analyze the potential $\Phi(r)$ in the plane of the ring ($z = 0$) for $r \ll R$ (point B in Figure 1). To the lowest non-zero power of r the electrostatic potential is given by $\Phi(r) \approx q(\alpha + \beta r^2)$.

B.1	Find the expression for β. You might need to use the Taylor expansion formula given above.	1.5pt

B.2	An electron is placed at point B (Figure 1, $r \ll R$). What is the force acting on the electron? Looking at the expression of the force, determine the sign of q so that the resulting motion would correspond to harmonic oscillations. The moving electron does not influence the charge distribution on the ring.	0.2pt

Part C. The focal length of the idealized electrostatic lens: instantaneous charging (2.3 points)

One wants to build a device to focus electrons—an electrostatic lens. Let us consider the following construction. The ring is situated perpendicularly to the z-axis, as shown in Figure 2. We have a source that produces on-demand packets of non-relativistic electrons. Kinetic energy of these electrons is $E = mv^2/2$ (v is velocity) and they leave the source at precisely controlled moments. The system is programmed so that the ring is charge-neutral most of the time, but its charge becomes q when electrons are closer than a distance $d/2$ ($d \ll R$) from the plane of the ring (shaded region in Figure 2, called “active region”). In part C assume that charging and de-charging processes are instantaneous and the electric field "fills the space" instantaneously as well. One can neglect magnetic fields and assume that the velocity of electrons in the z-direction is constant. Moving electrons do not perturb the charge distribution on the ring.

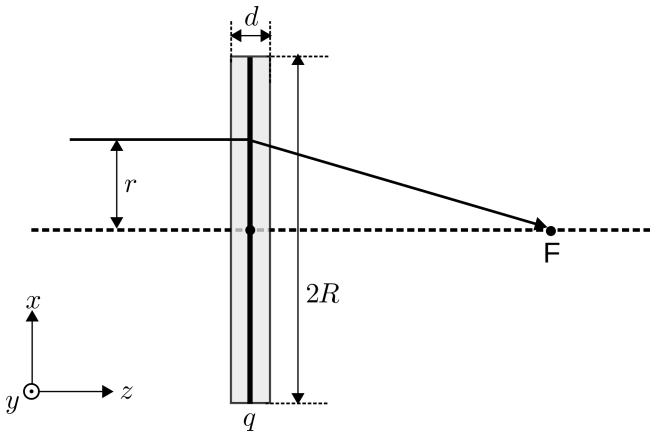

Figure 2. A model of an electrostatic lens.

C.1	Determine the focal length f of this lens. Assume that $f \gg d$. Express your answer in terms of the constant β from question B.1 and other known quantities. Assume that before reaching the "active region" the electron packet is parallel to the z-axis and $r \ll R$. The sign of q is such so that the lens is focusing.	1.3pt

In reality the electron source is placed on the z-axis at a distance $b > f$ from the center of the ring. Consider that electrons are no longer parallel to the z-axis before reaching the "active region", but are emitted from a point source at a range of different angles $\gamma \ll 1$ rad to the z-axis. Electrons are focused in a point situated at a distance c from the center of the ring.

C.2	Find c. Express your answer in terms of the constant β from question B.1 and other known quantities.	0.8pt

C.3 Is the equation of a thin optical lens 0.2pt

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{f}$$

fulfilled for the electrostatic lens? Show it by explicitly calculating $1/b+1/c$.

Part D. The ring as a capacitor (3 points)

The model of considered above was idealized and we assumed that the ring charged instantaneously. In reality charging is non-instantaneous, as the ring is a capacitor with a finite capacitance C. In this part we will analyze the properties of this capacitor. You might need the following integrals:

$$\int \frac{\mathrm{d}x}{\sin x} = -\ln\left|\frac{\cos x+1}{\sin x}\right| + \text{const}$$

and

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{1+x^2}\right| + \text{const}.$$

D.1 Calculate the capacitance C of the ring. Consider that the ring has a finite width $2a$, but remember that $a \ll R$. 2.0pt

When electrons reach the "active region", the ring is connected to a source of voltage V_0 (Figure 3). When electrons pass the "active region", the ring is connected to the ground. The resistance of contacts is R_0 and the resistance of the ring itself can be neglected.

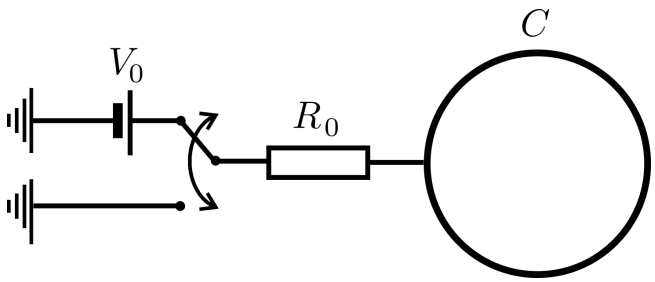

Figure 3. Charging of the electrostatic lens.

D.2 Determine the dependence of the charge on the ring as a function of time, $q(t)$, and make a schematic plot of this dependence. $t = 0$ corresponds to a time moment when electrons are in the plane of the ring. What is the charge on the ring q_0 when the absolute value $q(t)$ is maximum? The capacitance of the ring is C (i.e., you do not have to use the actual expression found in D.1). 1.0pt
Remark: the drawn polarity in Figure 3 is for indicative purposes only. The sign should be chosen so that the lens is focusing.

Part E. Focal length of a more realistic lens: non-instantaneous charging (2 points)

In this part of the problem, we will consider the action of this more realistic lens. Here we will again neglect the width of the ring $2a$ and will assume that electrons travel parallel to the z-axis before reaching the "active region". However, the charging of the ring is no longer instantaneous.

E.1	Find the focal length f of the lens. Assume that $f/v \gg R_0 C$, but d/v and $R_0 C$ are of the same order of magnitude. Express your answer in terms of the constant β from part B and other known quantities.	1.7pt

E.2	You will see, that the result for f is similar to that obtained in part C, whereby the value q is substituted with q_{eff}. Find the expression for q_{eff} in terms of quantities given in formulation of the problem.	0.3pt

Particles and Waves (10 points)

Wave–particle duality, which states that each particle can be described as a wave and vice versa, is one of the central concepts of quantum mechanics. In this problem, we will rely on this notion and just a few other basic assumptions to explore a selection of quantum phenomena covering the two distinct types of particles of the microworld—fermions and bosons.

Part A. Quantum particle in a box (1.4 points)

Consider a particle of mass m moving in a one-dimensional potential well, where its potential energy $V(x)$ is given by

$$V(x) = \begin{cases} 0, & 0 \le x \le L; \\ \infty, & x < 0 \text{ or } x > L. \end{cases} \tag{1}$$

While classical particle can move in such a potential having any kinetic energy, for quantum particle only some specific positive discrete energy levels are allowed. In any such allowed state, the particle can be described as a standing de Broglie wave with nodes at the walls.

A.1	Determine the minimal possible energy E_{min} of the quantum particle in the well. Express your answer in terms of m, L, and the Planck's constant h.	0.4pt

The particle's state with minimal possible energy is called the ground state, and all the rest allowed states are called excited states. Let us sort all the possible energy values in the increasing order and denote them as E_n, starting from E_1 for the ground state.

A.2	Find the general expression for the energy E_n (here $n = 1,\ 2,\ 3, \ldots$).	0.6pt

A.3	Particle can undergo instantaneous transition from one state to another only by emitting or absorbing a photon of the corresponding energy difference. Find the wavelength λ_{21} of the photon emitted during the transition of the particle from the first excited state (E_2) to the ground state (E_1).	0.4pt

Part B. Optical properties of molecules (2.1 points)

In this part, we will study several optical properties of the cyanine Cy5 molecule—a widely used dye molecule, schematically shown in Fig. 1a. Its optical properties are determined mainly by the carbon backbone, composed of the alternating single and double bonds between carbon atoms, shown in Fig. 1b, while the influence of the rings at the molecule's ends as well as radicals R is much smaller. Three of the four valence electrons of each C atom (and of N atoms) in the backbone form the chemical bonds, while the remaining valence electrons are "shared" and can move along the whole backbone. The net potential energy of each such electron is shown with oscillating thin line in Fig. 1c, with minima corresponding to the positions of the C and N atoms.

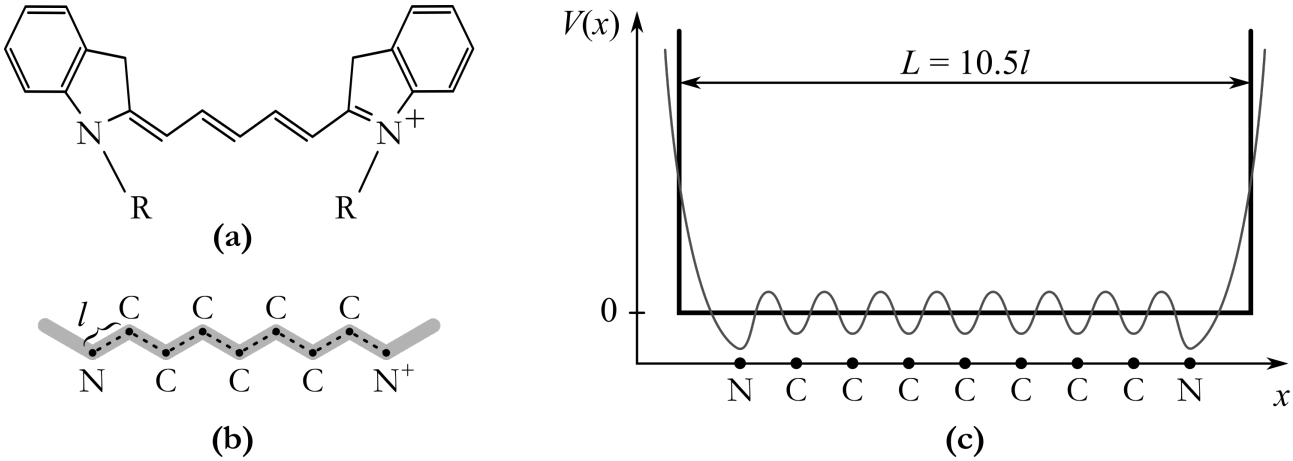

Figure 1. (a) Chemical structure of the cyanine Cy5 molecule (for simplicity, hydrogen atoms are not shown, and R denote some radicals). (b) The backbone of the Cy5 molecule, with mean inter-atomic distance l . (c) Potential energy of the electron along the backbone (thin line) and its approximation by the step function given by Eq. 1 (thick line).

For simplicity, we will approximate this potential energy profile by a simple function given in Eq. 1 with the width $L = 10.5l$ (see thick line in Fig. 1c), here $l = 140$ pm is the mean inter-atomic distance (see also Fig. 1b). As a result, we obtain the "electronic gas" composed of 10 electrons (7 from C atoms, 2 from the N atom, and 1 from the N^+ ion), moving in a one-dimensional potential well discussed in part A. In our evaluation, we can neglect the mutual interaction of these electrons; however, we should account for the fact that electrons are fermions and thus obey the Pauli exclusion principle. We also neglect the influence of other electrons as well as motion of the nuclei.

B.1	Evaluate the largest wavelength λ of the photon that can be absorbed by the Cy5 molecule assuming that the electron system is initially in its ground state. Express your answer in terms of l , physical constants and some numerical prefactor, and calculate the numerical value.	0.8pt

B.2	Another dye molecule Cy3 has similar structure, but its backbone is shorter by 2 carbon atoms. Is its absorption spectrum shifted to the bluer or to the redder spectral region compared to the Cy5 molecule? Evaluate numerically the magnitude $\Delta\lambda$ of this spectral shift. You can assume that removing two carbon atoms doesn't change the molecule shape and only makes the backbone length shorter by two interatomic distances.	0.4pt

Being in the excited state, molecule can undergo a spontaneous transition to the ground state while emitting photon. The mean rate K of such events (i.e. the relative decrease of the molecules being in the excited state, $\mathrm{d}N/N$, over time $\mathrm{d}t$, $K = \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t}$) is determined by the wavelength λ of the emitted photon, the transition electrical dipole moment d (which is of the order of $d \simeq el$, here e is elementary charge) as well as vacuum permittivity ε_0 and Planck's constant h.

B.3	Using dimension analysis, determine the expression for the rate of spontaneous emission in terms of ε_0, h, λ, and d. The numerical prefactor for your expression is $k = \frac{16}{3}\pi^3$.	0.7pt

B.4	For Cy5 molecule, $d \approx 2.4\, el$. Evaluate the mean fluorescence lifetime of the lowest excited state of Cy5 molecule, τ_{Cy5}, which is reciprocal to the rate of its emissive transition to the ground state.	0.2pt

Part C. Bose-Einstein condensation (1.5 points)

This part is not directly related to Parts A and B. Here, we will study the collective behaviour of bosonic particles. Bosons do not respect the Pauli exclusion principle, and—at low temperatures or high densities—experience a dramatic phenomenon known as the Bose–Einstein condensation (BEC). This is a phase transition to an intriguing collective quantum state: a large number of identical particles 'condense' into a single quantum state and start behaving as a single wave. The transition is typically reached by cooling a fixed number of particles below the critical temperature. In principle, it can also be induced by keeping the temperature fixed and driving the particle density past its critical value.

We begin by exploring the relation between the temperature and the particle density at the transition. As it turns out, estimates of their critical values can be deduced from a simple observation: *Bose-Einstein condensation takes place when the de Broglie wavelength corresponding to the mean square speed of the particles is equal to the characteristic distance between the particles in a gas*.

C.1	Given a non-interacting gas of ^{87}Rb atoms in thermal equilibrium, write the expressions for their typical linear momentum p and the typical de Broglie wavelength λ_{dB} as a function of atom's mass m, temperature T and physical constants.	0.4pt

C.2	Calculate the typical distance between the particles in a gas, ℓ, as a function of particle density n. Hence deduce the critical temperature T_c as a function of atom's mass, their density and physical constants.	0.5pt

To realize BEC in the lab, the experimentalists have to cool gases to temperatures as low as T_c = 100 nK.

C.3	What is the particle density of the Rb gas n_c if the transition takes place at such a temperature? For the sake of comparison, calculate also the 'ordinary' particle density n_0 of an ideal gas at the standard temperature and pressure (STP), i.e. $T_0 = 300$ K and $p_0 = 10^5$ Pa. How many times is the 'ordinary' gas denser? You may assume that the mass of the atoms is equal to 87 atomic mass units (m_{amu}).	0.6pt

Part D. Three-beam optical lattices (5 points)

The first Bose-Einstein condensates were produced back in 1995, and since then the experimental work has branched out in diverse directions. In this part, you will investigate one particularly fruitful idea to load the condensate into spatially periodic potentials created by interfering a number of coherent laser beams. Due to the periodic nature of the resulting interference patterns, they are referred to as *optical lattices*. The potential energy $V(\vec{r})$ of an atom moving in an optical lattice is proportional to the local intensity of the light, and in your calculations you may assume that

$$V(\vec{r}) = -\alpha \left\langle |\vec{E}(\vec{r}, t)|^2 \right\rangle. \tag{2}$$

Here, α is a *positive* constant, and the angle brackets indicate time-averaging which eliminates the rapidly

oscillating terms. The electric field produced by the *i*-th laser is described by

$$\vec{E}_i = E_{0,i}\,\vec{\varepsilon}_i \cos(\vec{k}_i \cdot \vec{r} - \omega t), \tag{3}$$

with the amplitude $E_{0,i}$, the wave vector $\vec{k}_i$, and the unit-length polarization vector $\vec{\varepsilon}_i$.

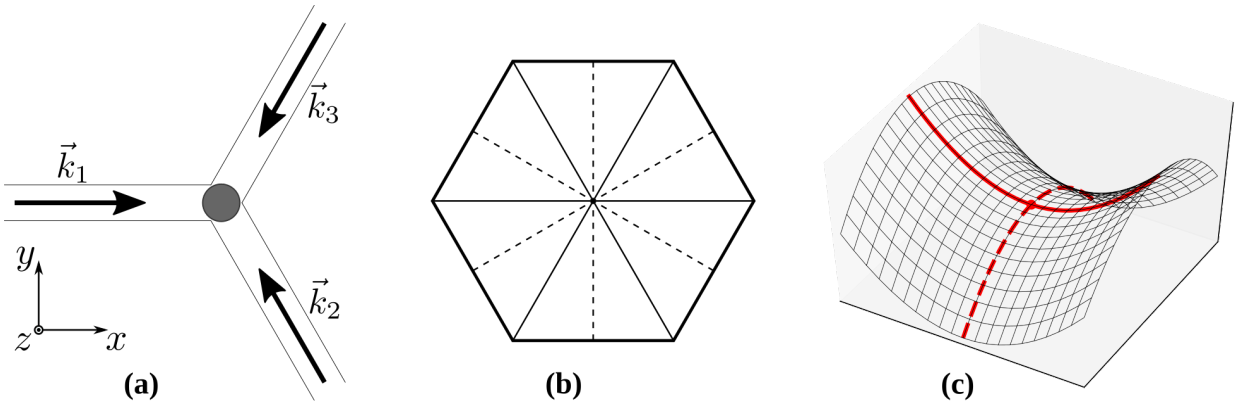

Figure 2. (a) Three-beam optical lattice: three plane waves with wave vectors $\vec{k}_{1,2,3}$ intersect and interfere in the area indicated by the grey circle. (b) Symmetries of a regular hexagon: solid and dashed lines show two sets of symmetry axes. (c) Saddle point: a point on a surface where the slopes in orthogonal directions are all zero, but which is not a local extremum of the plotted function. Travelling along the trajectory marked by the full line one encounters an apparent minimum. Additional analysis of the perpendicular direction (dashed line) is needed to distinguish a true minimum from a saddle point (shown).

Your task is to study *triangular optical lattices* that are produced by interfering three coherent laser beams of equal intensity. A typical setup is shown in Fig. 2a. Here, all three beams are polarized in the z direction, propagate in the xy plane and intersect at equal angles of 120°. Choose the direction of the x axis parallel to the wave vector $\vec{k}_1$.

D.1	Using Eqs. 2 and 3 obtain the expression for the potential energy $V(\vec{r})$ as a function of $\vec{r} = (x, y)$ in the plane of the beams. *Hint*: the result can be neatly expressed as a constant term plus a sum of three cosine functions of arguments $\vec{b}_i \cdot \vec{r}$. Please write your result in this form and identify the vectors $\vec{b}_i$.	1.4pt

D.2	The resulting potential energy has a sixfold rotational symmetry axis, i.e., the potential distribution is invariant with respect to a rotation by a multiple of 60° around the origin. Provide a simple argument to prove that this is indeed the case.	0.5pt

The above observation of symmetry simplifies the analysis of the two-dimensional potential distribution $V(\vec{r})$. As shown in Fig. 2b, a regular hexagon has symmetry lines that, respectively, connect opposite vertices (solid lines) and midpoints of opposite edges (dashed lines). Therefore, in our situation one does not need to produce and study two-dimensional potential plots as many insights can be deduced by focusing on the coordinate axes x and y that run along the symmetry lines.

D.3	Derive the behavior of the potential $V(\vec{r})$ along the coordinate axes, i.e., determine the functions $V_X(x) \equiv V(x,0)$ and $V_Y(y) \equiv V(0,y)$. Identify the locations of the extrema of $V_X(x)$ and $V_Y(y)$ as functions of a single argument. As these functions are periodic, please include in your lists only one representative from each family of periodically repeated minima and maxima.	1.2pt

We are interested in determining the locations of so-called *lattice sites*, i.e., the minima of the full two-dimensional potential $V(\vec{r})$. The obtained minima of single-argument functions V_X and V_Y identify their suspected positions but still have to be checked to eliminate the saddle points. As shown in Fig. 2c, when studied along a single line, saddle points may disguise as minima but they are not.

D.4	Review your results in the previous question to determine actual minima of the optical lattice: Identify all equivalent minima nearest to (but not coinciding with) the origin. What is the distance a between the nearest minima, in other words—the *lattice constant* of our optical lattice? Express the answer in terms of the laser wavelength λ_{las}.	0.8pt

Charge neutrality of ultracold atoms suggests that their interactions become relevant only when two or more atoms occupy the same site of an optical lattice. However, experimentalists are also able to explore setups that sustain long-range atomic interactions. A possible approach relies on creation of the so-called *Rydberg atoms* that are physically large and feature other exaggerated properties. Rydberg atoms are excited atoms with one electron promoted to a state with a very high principal quantum number n. The size of a Rydberg atom can be estimated by calculating the radius of the classical circular orbit of that electron with the orbital angular momentum $n\hbar$, here $\hbar$ is the reduced Planck constant.

D.5	Calculate the value of n that corresponds to the radius of the Rb Rydberg atom comparable to the wavelength of laser light λ_{las} = 380 nm. Give your answer in terms of λ_{las} and physical constants and find its numerical value.	1.1pt