Planetary Physics (10 points)

Part A. Mid-ocean ridge (5.0 points)

A.1 (0.8 points)

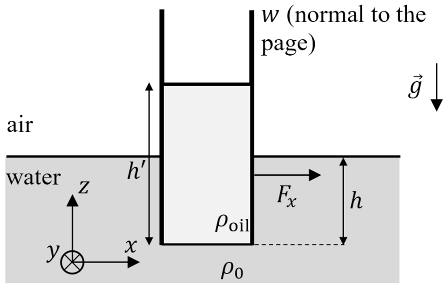

Figure 1

Let h' be the height of the column of oil (see Fig. 1). Then pressure at depth h below the water surface must be $p_h = \rho_{oil} gh = \rho_{oil} gh'$, from where $h' = \frac{\rho_0}{\rho_{oil}} h$. Horizontal force on the plate $F_x = F_1 - F_0$, where the force due to new fluid is $F_1 = \frac{\rho_{oil} gh'}{2} \cdot h'w$ and the force due to water is $F_0 = \frac{\rho_0 gh}{2} \cdot hw$.

Combining all the equation above, we get

$$F_x = \left(\frac{\rho_0}{\rho_{oil}} - 1\right) \frac{\rho_0 gh^2 w}{2}.$$

A.1 (0.8 pt)

$$F_x = \left(\frac{\rho_0}{\rho_{oil}} - 1\right) \frac{\rho_0 gh^2 w}{2}.$$

A.2 (0.6 points)

Consider a rectangular mass element of the crust. Since relation $l(T) = l_1 \left[1 - k_l \left(T_1 - T\right) / (T_1 - T_0)\right]$ holds for all three dimensions of the solid, its volume V satisfies

$$V = V_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^3,$$

where V_1 is the volume at $T = T_1$. If the mass of the element is m, density is then

$$\rho(T) = \frac{m}{V} = \frac{m}{V_1} \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3} = \rho_1 \left(1 - k_l \frac{T_1 - T}{T_1 - T_0}\right)^{-3}.$$

Since $k_l \ll 1$, this can be approximated as

$$\rho(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right),$$

so that $k = 3k_l$.

A.2 (0.6 pt)

$\rho(T) \approx \rho_1 \left(1 + 3k_l \frac{T_1 - T}{T_1 - T_0}\right)$. $\quad k = 3k_l$.

A.3 (1.1 points)

Since mantle behaves like a fluid in hydrostatic equilibrium, pressure $p(x,z)$ at $z = h + D$ must be the same for all x. Therefore,

$$p(0, h+D) = p(\infty, h+D).$$

Similarly, we must have

$$p(0,0) = p(\infty, 0).$$

Hence, the change in pressure between $z = 0$ and $z = \infty$ must be the same at both $x = 0$ and $x = \infty$. At the ridge axis

$$p(0, h+D) - p(0,0) = \rho_1 g (h+D),$$

while far away

$$p(\infty, h+D) - p(\infty, 0) = \rho_0 g h + \int_h^{h+D} \rho(T(\infty, z))\, g\, \mathrm{d}z.$$

Far away from the ridge axis the two surfaces of the crust are effectively horizontal, meaning that the law of heat conduction can be written as

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \text{const.}$$

Hence, after applying the relevant temperature boundary conditions,

$$T(\infty, z) = T_0 + (T_1 - T_0)\frac{z-h}{D}.$$

From all the equations above and by using the density formula given in the problem text,

$$\rho_1 g (h+D) = \rho_0 g h + \int_h^{h+D} \rho_1 \left(1 + k \frac{T_1 - T_0 - (T_1 - T_0)\frac{z-h}{D}}{T_1 - T_0}\right) g\, \mathrm{d}z,$$

from where we straightforwardly obtain

$$D = \frac{2}{k}\left(1 - \frac{\rho_0}{\rho_1}\right) h.$$

A.3 (1.1 pt)

$$D = \frac{2}{k}\left(1 - \frac{\rho_0}{\rho_1}\right) h.$$

A.4 (1.6 points)

The net horizontal force on the half of the ridge is the difference between the pressure forces acting at $x = 0$ and $x = \infty$:

$$F = L\int_0^{h+D} \left(p\left(0, z\right) - p\left(\infty, z\right)\right)\, \mathrm{d}z.$$

From considerations of the previous question, pressure at $x = 0$ is

$$p\left(0, z\right) = p\left(0, 0\right) + \rho_1 g z,$$

while very far away

$$p\left(\infty, z\right) = \begin{cases} p\left(\infty, 0\right) + \rho_0 g z & \text{if } 0 \le z \le h, \\ p\left(\infty, 0\right) + \rho_0 g h + \int_h^z \rho_1 \left(1 + k\frac{T_1 - T_0 - (T_1 - T_0)\frac{z'-h}{D}}{T_1 - T_0}\right) g\, \mathrm{d}z' & \text{if } h \le z \le h + D. \end{cases}$$

The equations above can be combined into

$$F = L\int_0^{h+D} \left(p\left(0, 0\right) + \rho_1 g z\right)\, \mathrm{d}z - L\int_0^{h} \left(p\left(\infty, 0\right) + \rho_0 g z\right)\, \mathrm{d}z -$$
$$- L\int_h^{h+D} \left(p\left(\infty, 0\right) + \rho_0 g h\right)\, \mathrm{d}z - L\int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k\left(1 - \frac{z'-h}{D}\right)\right) g\, \mathrm{d}z'\right] \mathrm{d}z.$$

The double integral can be easily found either directly or by using a substitution $u = z - h$, $u' = z' - h$:

$$\int_h^{h+D} \left[\int_h^z \rho_1 \left(1 + k\left(1 - \frac{z'-h}{D}\right)\right) g\, \mathrm{d}z'\right] \mathrm{d}z = \int_0^D \left[\int_0^u \rho_1 \left(1 + k\left(1 - \frac{u}{D}\right)\right) g\, \mathrm{d}u'\right] \mathrm{d}u$$

After a straightforward integration and using $p\left(0, 0\right) = p\left(\infty, 0\right)$ as well as the result of the previous question,

$$F = gL\left[\rho_1\left(\frac{h^2}{2} + hD - \frac{kD^2}{3}\right) - \rho_0\left(\frac{h^2}{2} + hD\right)\right] = gLh^2\left(\rho_1 - \rho_0\right)\left(\frac{1}{2} + \frac{2}{3k}\left(1 - \frac{\rho_0}{\rho_1}\right)\right).$$

Since $k \ll 1$, the term with $\frac{1}{k}$ is of the leading order, hence, the required answer is

$$F \approx \frac{2gLh^2}{3k}\frac{\left(\rho_1 - \rho_0\right)^2}{\rho_1}.$$

A.4 (1.6 pt)

$$F \approx \frac{2gLh^2}{3k}\frac{(\rho_1-\rho_0)^2}{\rho_1}.$$

A.5 (0.9 points)

The timescale τ is expected to depend only on density of the crust ρ_1, its specific heat c, thermal conductivity κ and thickness D. Hence, we can write $\tau = A\rho_1^\alpha c^\beta \kappa^\gamma D^\delta$, where A is a dimensionless constant. We will obtain the powers α–δ via dimensional analysis.

Define the symbols for different dimensions: L for length, M for mass, T for time and Θ for temperature. Then τ, ρ_1, c, κ and D have dimensions T, $\mathrm{ML^{-3}}$, $\mathrm{L^2T^{-2}\Theta^{-1}}$, $\mathrm{MLT^{-3}\Theta^{-1}}$ and L, respectively. The resulting set of linear equations to balance the powers of length, mass, time and temperature, respectively, is

$$\begin{cases} 0 = -3\alpha + 2\beta + \gamma + \delta, \\ 0 = \alpha + \gamma, \\ 1 = -2\beta - 3\gamma, \\ 0 = -\beta - \gamma. \end{cases}$$

This gives $\alpha = \beta = 1$, $\gamma = -1$, $\delta = 2$. Hence,

$$\tau = A\frac{c\rho_1 D^2}{\kappa}.$$

A.5 (0.9 pt)

$$\tau \approx \frac{c\rho_1 D^2}{\kappa}.$$

Part B. Seismic waves in a stratified medium (5.0 points)

B.1 (1.5 points)

Seismic waves in this problem can be treated by using ray theory. Namely, their propagation is described by the Snell's law of refraction

$$n(0)\sin\theta_0 = n(z)\sin\theta,$$

where the refractive index is

$$n(z) = \frac{c}{v(z)} = \frac{c}{v_0\left(1+\frac{z}{z_0}\right)}$$

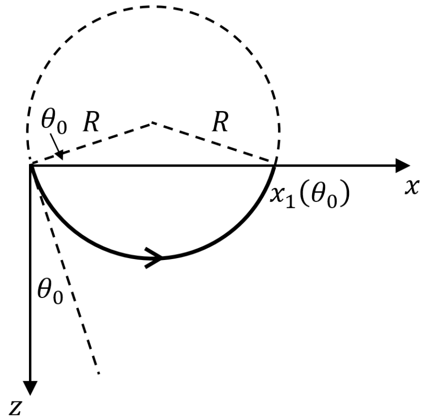

Figure 2

and c denotes the seismic wave speed in a material with refractive index $n = 1$. From the two equations above we have

$$v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta.$$

Method 1. Since this describes an arc of a circle, we have that at $\theta = \frac{\pi}{2}$, $z = R - R\sin\theta_0$ (fig. 2), giving

$$\left(1+\frac{R-R\sin\theta_0}{z_0}\right)\sin\theta_0 = 1,$$

from where the circle radius $R = \frac{z_0}{\sin\theta_0}$. From simple geometry we get

$$x_1\left(\theta_0\right) = 2R\cos\theta_0,$$

i.e. $A = 2z_0$ and $b = 1$.

Method 2. Implicitly differentiating $v_0\left(1+\frac{z}{z_0}\right)\sin\theta_0 = v_0\sin\theta$ gives

$$\frac{\mathrm{d}z}{z_0}\sin\theta_0 = \cos\theta\,\mathrm{d}\theta.$$

An infinitesimal ray path length $\mathrm{d}l$ is related to the change in the vertical coordinate via

$$\mathrm{d}z = \mathrm{d}l\,\cos\theta,$$

giving

$$\mathrm{d}l = \frac{z_0}{\sin\theta_0}\,\mathrm{d}\theta.$$

This is an equation of an arc of a circle of radius $R = \frac{z_0}{\sin\theta_0}$

Alternatively, instead of considering an infinitesimal ray path length $\mathrm{d}l$, one can obtain the answer by writing

$$\cot\theta = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}.$$

The first derivative can be eliminated via Snell's law, leading to

$$\cot\theta = \frac{z_0 \cos\theta}{\sin\theta_0}\frac{\mathrm{d}\theta}{\mathrm{d}x},$$

which can be integrated to get

$$x_1 = -\frac{z_0}{\sin\theta_0}\int_{\text{start}}^{\text{end}} \mathrm{d}\cos\theta = \frac{2z_0\cos\theta_0}{\sin\theta_0},$$

where we used Snell's law again to get that the ray has $\cos\theta = -\cos\theta_0$ at the point where it reaches the surface.

B.1 (1.5 pt)

$x_1(\theta_0) = 2z_0\cot\theta_0.$

B.2 (1.5 points)

In two dimensions, $\frac{E}{\pi}\,\mathrm{d}\theta_0$ is the energy carried by rays that are emitted within interval $[\theta_0, \theta_0 + \mathrm{d}\theta_0)$. On the other hand, the energy carried by rays that arrive at $[x, x + \mathrm{d}x)$ is $\varepsilon\,\mathrm{d}x$. Therefore,

$$\varepsilon = \frac{E}{\pi}\left|\frac{\mathrm{d}\theta_0}{\mathrm{d}x}\right|.$$

Using the result of question B.1,

$$\frac{\mathrm{d}x}{\mathrm{d}\theta_0} = -\frac{Ab}{\sin^2(b\theta_0)} = -Ab\left(1+\cot^2(b\theta_0)\right) = -\frac{b\left(A^2+x^2\right)}{A}.$$

Hence,

$$\varepsilon(x) = \frac{EA}{\pi b\left(A^2+x^2\right)} = \frac{2Ez_0}{\pi\left(4z_0^2+x^2\right)}.$$

This function is plotted in Fig. 3.

B.2 (1.5 pt)

$\varepsilon(x) = \dfrac{EA}{\pi b\left(A^2+x^2\right)} = \dfrac{2Ez_0}{\pi\left(4z_0^2+x^2\right)}.$ Sketch is shown in Fig. 3.

B.3 (2.0 points)

Define $x_- = x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right)$ and $x_+ = x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)$. To the leading order in $\delta\theta_0$, $x_- \approx x_+ \approx x_1(\theta_0)$. With each reflection of the signal, the horizontal distance between the points where the edges of

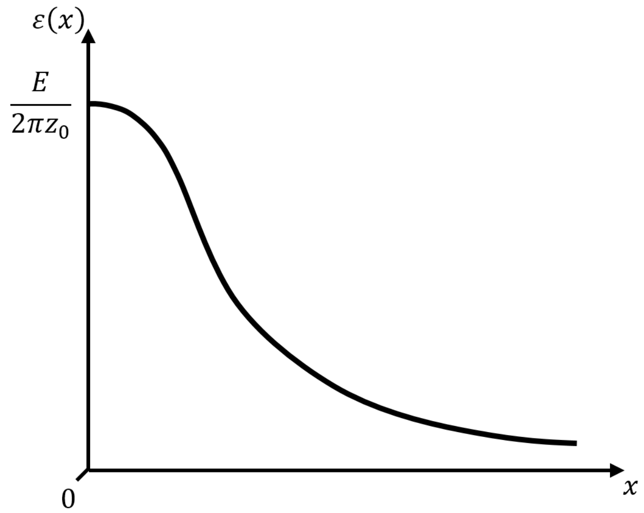

Figure 3. Plot of the function $\varepsilon(x)$.

the signal reflect increases by $|x_+ - x_-| = x_- - x_+$. When moving along the positive x-axis, these zones get wider until they overlap. If this happens after N reflections, then

$$N \approx \frac{x_1(\theta_0)}{x_- - x_+},$$

where the approximate sign tends to equality as $\delta\theta_0 \to 0$.

The position where the zones start to overlap is at $x_{\mathrm{max}} = N x_1(\theta_0)$. Therefore,

$$x_{\mathrm{max}} = \frac{x_1(\theta_0)^2}{x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right)}.$$

Since $\delta\theta_0 \ll \theta_0$, we can approximate

$$x_1\left(\theta_0 - \frac{\delta\theta_0}{2}\right) - x_1\left(\theta_0 + \frac{\delta\theta_0}{2}\right) \approx -\frac{\mathrm{d}x_1(\theta_0)}{\mathrm{d}\theta_0}\delta\theta_0 = \frac{Ab}{\sin^2(b\theta_0)}\delta\theta_0.$$

Combining the last two equations and substituting the $x_1(\theta_0)$ expression gives

$$x_{\mathrm{max}} = \frac{Ab\cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}.$$

B.3 (2.0 pt)

$$x_{\mathrm{max}} = \frac{Ab\cos^2(b\theta_0)}{\delta\theta_0} = \frac{2z_0\cos^2\theta_0}{\delta\theta_0}.$$

Electrostatic lens (10 points)

Part A. Electrostatic potential on the axis of the ring (1 point)

A.1 (0.3 points)

The linear charge density of the ring is $\lambda = q/(2\pi R)$. All the points of the ring are situated a distance $\sqrt{R^2+z^2}$ away from point A. Integrating over the whole ring we readily obtain:

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}.$$

A.1 (0.3 pt)

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}}.$$

A.2 (0.4 points)

Using an expansion in powers of z we obtain:

$$\Phi(z) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{R^2+z^2}} = \frac{q}{4\pi\varepsilon_0 R}\frac{1}{\sqrt{1+\left(\frac{z}{R}\right)^2}} \approx \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right).$$

A.2 (0.4 pt)

$$\Phi(z) \approx \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right).$$

A.3 (0.2 points)

The potential energy of the electron is $V(z) = -e\Phi(z)$. The force acting on the electron is

$$F(z) = -\frac{\mathrm{d}V(z)}{\mathrm{d}z} = +e\frac{\mathrm{d}\Phi}{\mathrm{d}z} = -\frac{qe}{4\pi\varepsilon_0 R^3}z.$$

If this is a restoring force, it should be negative for positive z. Thus, $q>0$.

A.3 (0.2 pt)

$$F(z) = -\frac{qe}{4\pi\varepsilon_0 R^3}z. \qquad q>0.$$

A.4 (0.1 points)

The equation of motion for an electron is

$$m\ddot{z} + \frac{qe}{4\pi\varepsilon_0 R^3} z = 0$$

(here dots denote time derivatives). We therefore get

$$\omega = \sqrt{\frac{qe}{4\pi m\varepsilon_0 R^3}}.$$

A.4 (0.1 pt)

$$\omega = \sqrt{\frac{qe}{4\pi m\varepsilon_0 R^3}}.$$

Part B. Electrostatic potential in the plane of the ring (1.7 points)

B.1 (1.5 points)

There are two different ways to solve this problem: (i) using direct integration; (ii) using Gauss's law and the result of part A.

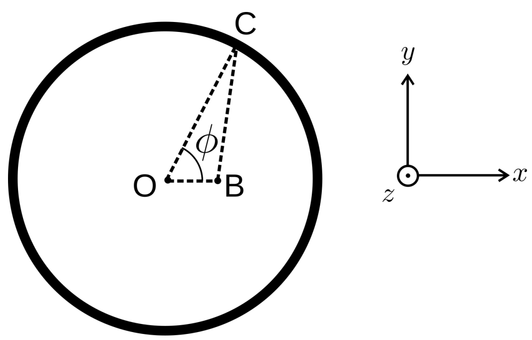

Figure 1: Calculating electrostatic potential in the plane of the ring through direct integration.

(i) **Direct integration**. We will follow the notations of Figure 1. Since the potential has cylindrical symmetry, let the point B, where we calculate the potential, be on the x-axis. Let

$$|\mathrm{OB}| = r; |\mathrm{OC}| = R.$$

Thus:

$$|\mathrm{BC}|^2 = R^2 + r^2 - 2Rr\cos\phi.$$

Electrostatic potential created by ring element $\mathrm{d}\phi$ at the point B:

$$\mathrm{d}\Phi = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,\mathrm{d}\phi}{\sqrt{R^2+r^2-2Rr\cos\phi}} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\,\mathrm{d}\phi}{\sqrt{1+\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi}}.$$

Using the expansion given in the formulation of the problem for $\varepsilon = -1/2$ we have:

$$\mathrm{d}\Phi \approx \frac{\lambda\,\mathrm{d}\phi}{4\pi\varepsilon_0}\left[1-\frac{1}{2}\left(\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)+\frac{3}{8}\left(\frac{r^2}{R^2}-2\frac{r}{R}\cos\phi\right)^2\right].$$

Ignoring the terms of the order r^3 and r^4 we get:

$$\mathrm{d}\Phi \approx \frac{\lambda\,\mathrm{d}\phi}{4\pi\varepsilon_0}\left[1+\frac{r}{R}\cos\phi+\frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\phi-\frac{1}{2}\right)\right].$$

Integrating over all angles we finally obtain:

$$\Phi(r) = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\left[1+\frac{r}{R}\cos\phi+\frac{r^2}{R^2}\left(\frac{3}{2}\cos^2\phi-\frac{1}{2}\right)\right]\mathrm{d}\phi.$$

$$\Phi(r) = \frac{q}{4\pi\varepsilon_0 R}\left(1+\frac{r^2}{4R^2}\right).$$

From here, comparing with the expression $\Phi(r) = q(\alpha+\beta r^2)$, we obtain

$$\beta = \frac{1}{16\pi\varepsilon_0 R^3}.$$

(ii) **Gauss's law.**

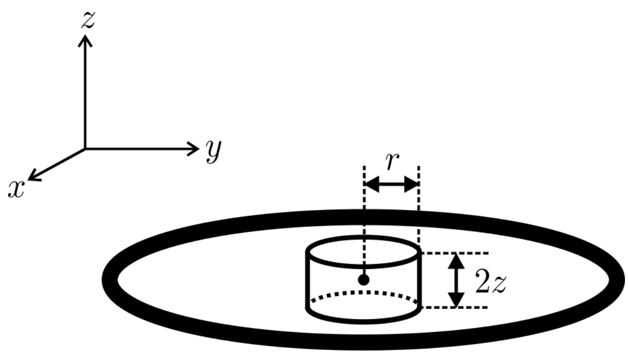

Figure 2: Calculating electrostatic potential in the plane of the ring via Gauss's law.

Let us analyze a small cylinder of radius r. The center of the cylinder coincides with the center of the ring. In part A we analyzed the potential along the z-axis, while in this part we analyze the potential along the radius r. For any $z \ll R$ and $r \ll R$ the potential has an expression:

$$\Phi(z,r) = \frac{q}{4\pi\varepsilon_0 R}\left(1-\frac{z^2}{2R^2}\right)+q\beta r^2.$$

The lowest order terms are quadratic in r and z. Due to reflection symmetry the potential does not contain terms of the type rz. This, for example, immediately gives us $\alpha = 1/(4\pi\varepsilon_0 R)$. Thus, for small r and z electric fields in the radial and axial directions are:

$$\mathcal{E}_z(z,r) = +\frac{q}{4\pi\varepsilon_0 R^3}z, \qquad \mathcal{E}_r(z,r) = -2q\beta r.$$

Applying Gauss's law to the cylinder we obtain:

$$\oint \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 0 \qquad \Rightarrow \qquad \int_{\text{side}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} + \int_{\text{base}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 0.$$

The second integral is:

$$\int_{\text{base}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 2\pi r^2 \mathcal{E}_z(z,r) = \frac{qzr^2}{2\varepsilon_0 R^3}.$$

The first integral is:

$$\int_{\text{side}} \vec{\mathcal{E}} \cdot \mathrm{d}\vec{S} = 4\pi rz \mathcal{E}_r(z,r) = -8\pi q\beta r^2 z.$$

Gauss's theorem thus gives:

$$\frac{qzr^2}{2\varepsilon_0 R^3} - 8\pi q\beta r^2 z = 0.$$

This immediately yields

$$\beta = \frac{1}{16\pi\varepsilon_0 R^3},$$

which agrees with the result obtained via direct integration.

B.1 (1.5 pt)

$$\beta = \frac{1}{16\pi\varepsilon_0 R^3}.$$

B.2 (0.2 points)

The potential of the electron is $V(r) = -e\Phi(r)$. Force acting on the electron in the xy plane is

$$F(r) = -\frac{\mathrm{d}V(r)}{\mathrm{d}r} = +e\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = \frac{qe}{8\pi\varepsilon_0 R^3}r.$$

To have oscillations we need the force to be negative for $r > 0$. Thus, $q < 0$.

B.2 (0.2 pt)

$$F(r) = +\frac{qe}{8\pi\varepsilon_0 R^3}r. \qquad\qquad q < 0.$$

Part C. The focal length of the idealized electrostatic lens (2.3 points)

C.1 (1.3 points)

Let us consider an electron with the velocity $v = \sqrt{2E/m}$ at a distance r from the "optical" axis (Figure 2 of the problem). The electron crosses the "active region" of the lens in time

$$t = \frac{d}{v}.$$

The equation of motion in the r direction:

$$m\ddot{r} = 2eq\beta r.$$

During the time the electron crosses the active region of the lens, the electron acquires radial velocity:

$$v_r = \frac{2eq\beta r}{m}\frac{d}{v} < 0.$$

The lens will be focusing if $q < 0$. The time it takes for an electron to reach the "optical" axis is:

$$t' = \frac{r}{|v_r|} = -\frac{mv}{2eq\beta d}.$$

During this time the electron travels in the z-direction a distance

$$\Delta z = t'v = -\frac{mv^2}{2eq\beta d} = -\frac{E}{eqd\beta}.$$

Δz does not depend on the radial distance r, therefore all electron will cross the "optical" axis (will be focused) in the same spot. Thus,

$$f = -\frac{E}{eqd\beta}.$$

C.1 (1.3 pt)

$$f = -\frac{E}{eqd\beta}.$$

C.2 (0.8 points)

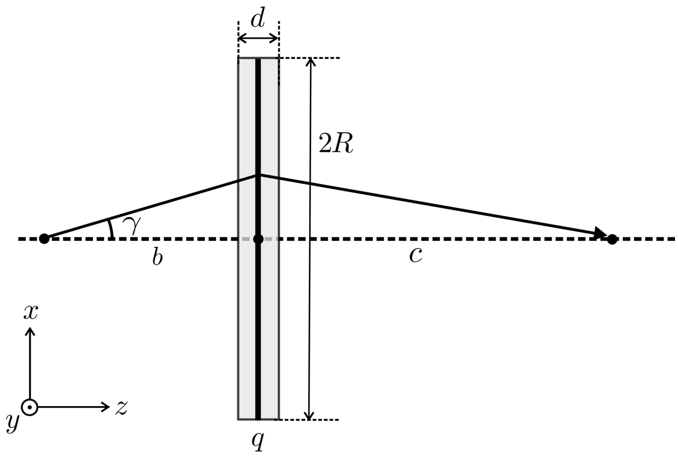

Figure 3: Focusing of electrons.

Let us consider an electron emitted an an angle γ to the optical axis (Figure 3). Its initial velocity in the radial direction is:

$$v_{r;0} = v\sin\gamma \approx v\gamma \approx v\frac{r}{b},$$

where r is the radial distance of the electron when it reaches the plane of the ring. The velocity in the z-direction is

$$v_z = v\cos\gamma \approx v.$$

For small angles γ the additional velocity in the r-direction acquired in the "active region" is the same as in part C.1. Thus, the radial velocity after crossing the active region is

$$v_r = v\frac{r}{b} + \frac{2eq\beta r}{m}\frac{d}{v},$$

where the first term is positive and the second term is negative, since $q < 0$. If the electrons are focused, then $v_r < 0$ (this can be verified after obtaining the final result). The electron will reach the optical axis in time

$$t' = \frac{r}{|v_r|} = -\frac{r}{\frac{2eq\beta r}{m}\frac{d}{v} + v\frac{r}{b}} = -\frac{1}{\frac{2eq\beta}{m}\frac{d}{v} + \frac{v}{b}}.$$

During this time it will travel a distance

$$c = t'v = -\frac{1}{\frac{2eq\beta}{m}\frac{d}{v^2} + \frac{1}{b}} = -\frac{1}{\frac{eq\beta d}{E} + \frac{1}{b}}.$$

C.2 (0.8 pt)

$$c = -\frac{1}{\frac{eq\beta d}{E} + \frac{1}{b}}.$$

C.3 (0.2 pt)

From the previous answer we obtain:

$$\frac{1}{b}+\frac{1}{c}=-\frac{eq\beta d}{E}.$$

Comparing with the answer of C.1 we immediately obtain

$$\frac{1}{b}+\frac{1}{c}=\frac{1}{f},$$

i.e. the equation of a thin optical lens is valid for an electrostatic lens as well.

C.3 (0.2 pt)

The equation of a thin optical lens $\frac{1}{b}+\frac{1}{c}=\frac{1}{f}$ is **valid** for an electrostatic lens.

Part D. The ring as a capacitor (3 points)

D.1 (2.0 points)

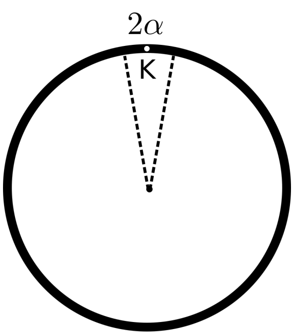

Figure 4: Calculation of the capacitance of the ring.

Let us sub-divide the entire ring into two parts: a part corresponding to the angle $2\alpha \ll 1$, and the rest of the ring, as shown in Figure 4. While the angle is small in comparison to 1, let us assume that the length of the first part, αR, is still large compared to a ($\alpha R \gg a$). Let us calculate the electrostatic potential Φ at point K. It it a sum of two terms: the first one produced by the cut-out part with an angle 2α (contribution Φ_1) and the second one originating from the rest of the ring (contribution Φ_2).

Contribution Φ_1. Since $\alpha \ll 1$, we can neglect the curvature of the cylinder that is cut out from the ring. The linear charge density on the ring is $\lambda=\frac{q}{2\pi R}$. The potential at the center of the cylinder is then given by an integral:

$$\Phi_1 = 2\frac{1}{4\pi\varepsilon_0}\frac{q}{2\pi R}\int_0^{\alpha R}\frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_0^{\alpha R}\frac{\mathrm{d}(x/a)}{\sqrt{1+(x/a)^2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_0^{\alpha R/a}\frac{\mathrm{d}y}{\sqrt{1+y^2}}.$$

Using the integral provided in the description of the problem we get:

$$\Phi_1 = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(y+\sqrt{1+y^2}\right)\Big|_0^{\alpha R/a} = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{\alpha R}{a}+\sqrt{1+\left(\frac{\alpha R}{a}\right)^2}\right).$$

As $\alpha R \gg a$,

$$\Phi_1 \approx \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{2\alpha R}{a}\right).$$

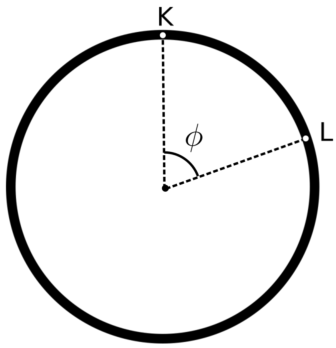

Figure 5: Calculation of the capacitance of the ring

Contribution Φ_2. In this case we can neglect the thickness a. Using the cosine theorem we can derive the distance between points K and L of Figure 5:

$$|\mathrm{KL}| = 2R\sin\frac{\phi}{2}.$$

The contribution Φ_2 can then be written as an integral:

$$\Phi_2 = 2\frac{q}{2\pi}\frac{1}{4\pi\varepsilon_0}\int_\alpha^\pi\frac{\mathrm{d}\phi}{2R\sin\frac{\phi}{2}} = \frac{q}{8\pi^2\varepsilon_0 R}\int_\alpha^\pi\frac{\mathrm{d}\phi}{\sin\frac{\phi}{2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_\alpha^\pi\frac{\mathrm{d}\left(\frac{\phi}{2}\right)}{\sin\frac{\phi}{2}} = \frac{q}{4\pi^2\varepsilon_0 R}\int_{\alpha/2}^{\pi/2}\frac{\mathrm{d}\chi}{\sin\chi}.$$

Using the integral from the formulation of the problem, we calculate:

$$\int_{\alpha/2}^{\pi/2}\frac{\mathrm{d}\chi}{\sin\chi} = -\ln\left(\frac{\cos\chi+1}{\sin\chi}\right)\Big|_{\alpha/2}^{\pi/2} = \ln\left(\frac{\cos\alpha/2+1}{\sin\alpha/2}\right) \approx \ln\left(\frac{4}{\alpha}\right)$$

for $\alpha \ll 1$. Therefore

$$\Phi_2 \approx \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{4}{\alpha}\right).$$

The total potential and capacitance. The total potential is the sum of Φ_1 and Φ_2:

$$\Phi = \Phi_1 + \Phi_2 = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{2\alpha R}{a}\right) + \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{4}{\alpha}\right) = \frac{q}{4\pi^2\varepsilon_0 R}\ln\left(\frac{8R}{a}\right).$$

α drops out from the expression. From here we obtain the capacitance $C = q/\Phi$:

$$C = \frac{4\pi^2\varepsilon_0 R}{\ln\left(\frac{8R}{a}\right)}.$$

$C \to 0$ as $a \to 0$.

D.1 (2.0 pt)

$$C = \frac{4\pi^2\varepsilon_0 R}{\ln\left(\frac{8R}{a}\right)}.$$

D.2 (1.0 point)

Let $q(t)$ be the charge on the ring at a time t. Potential of the disk is thus $q(t)/C$. Voltage drop of the resistor is $R_0 I(t) = R_0\,\mathrm{d}q/\mathrm{d}t$. Therefore for time $-\frac{d}{2v} < t < \frac{d}{2v}$:

$$\frac{q(t)}{C} + R_0\frac{\mathrm{d}q}{\mathrm{d}t} = V_0.$$

Integrating this equation and keeping in mind that $q(t) = 0$ at $t = -d/(2v)$, we get:

$$q(t) = CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right).$$

The charge attains the largest absolute value at $t = d/(2v)$. The value of the charge at this time is:

$$q_0 = CV_0\left(1 - \mathrm{e}^{-\frac{d}{vR_0C}}\right).$$

When $t > \frac{d}{2v}$, we get:

$$\frac{q(t)}{C} + R_0\frac{\mathrm{d}q}{\mathrm{d}t} = 0.$$

From here:

$$q(t) = q_0\mathrm{e}^{-\frac{t}{R_0C}+\frac{d}{2vR_0C}} = CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{RC}}.$$

Therefore, we obtain:

$$q(t) = \begin{cases} 0 & \text{for } t < -\frac{d}{2v}; \\ CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right) & \text{for } -\frac{d}{2v} < t < \frac{d}{2v}; \\ CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{R_0C}} & \text{for } t > \frac{d}{2v}. \end{cases}$$

For a lens to be focusing we require that charge is negative, therefore $V_0 < 0$. The dependence of charge on time is shown in Figure 6.

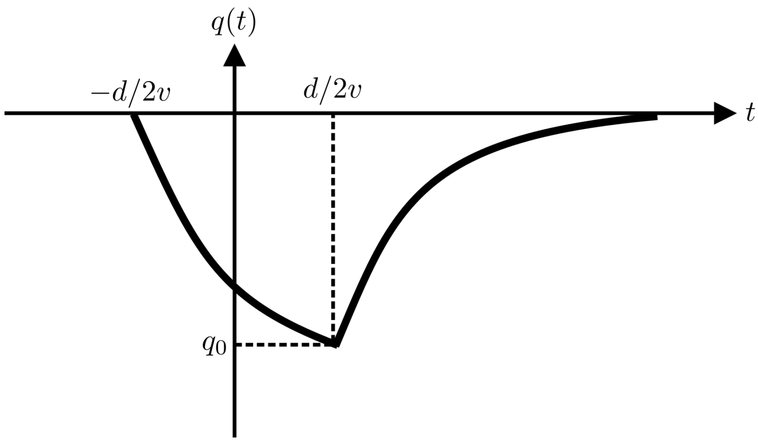

Figure 6: Charge on the ring as a function of time.

D.2 (1.0 pt)

For $-\dfrac{d}{2v} < t < \dfrac{d}{2v}$, $\quad q\,(t) = CV_0\left(1 - \mathrm{e}^{-\frac{d}{2vR_0C}}\mathrm{e}^{-\frac{t}{R_0C}}\right)$.

For $t > \dfrac{d}{2v}$, $\quad q\,(t) = CV_0\left(\mathrm{e}^{\frac{d}{2vR_0C}} - \mathrm{e}^{-\frac{d}{2vR_0C}}\right)\mathrm{e}^{-\frac{t}{R_0C}}$.

$q_0 = CV_0\left(1 - \mathrm{e}^{-\frac{d}{vR_0C}}\right)$. Schematic plot of this function is shown in Figure 6.

Part E. Focal length of a more realistic lens (2 points)

E.1 (1.7 points)

Like in part C, the radial equation of motion of an electron is:

$$m\ddot{r} = 2eq(t)\beta r,$$

where in this case $q(t)$ depends on time. Using the notation $\eta = 2e\beta/m$, we obtain:

$$\ddot{r} - \eta q(t) r = 0.$$

As $f/v \gg R_0C$, then during charging–decharging the electron does not substantially change its radial position r, and we can assume r to be constant during the entire charging–decharging process. In this case the acquired vertical velocity is

$$v_r = \eta r \int_{-d/(2v)}^{\infty} q(t)\,\mathrm{d}t.$$

We can use the derived equations for $q(t)$ and find the integrals. The integral $\int_{-d/(2v)}^{d/(2v)} q(t)\,\mathrm{d}t$ is (using the notation $d/v = t_0$, $R_0C = \tau$, $CV_0 = Q_0$):

$$\int_{-t_0/2}^{t_0/2} q(t)\,\mathrm{d}t = \int_{-t_0/2}^{t_0/2} Q_0 \left(1 - \mathrm{e}^{-\frac{t_0}{2\tau}} \mathrm{e}^{-\frac{t}{\tau}}\right) \mathrm{d}t = Q_0 \left(t_0 - \tau \left[1 - \mathrm{e}^{-t_0/\tau}\right]\right).$$

The integral $\int_{d/(2v)}^{\infty} q(t)\,\mathrm{d}t$ is

$$\int_{t_0/2}^{\infty} Q_0 \left(e^{\frac{t_0}{2\tau}} - e^{-\frac{t_0}{2\tau}}\right) \mathrm{e}^{-\frac{t}{\tau}}\,\mathrm{d}t = Q_0\tau \left[1 - \mathrm{e}^{-t_0/\tau}\right].$$

Adding the two integrals we obtain for the final integral:

$$\int_{-t_0/2}^{\infty} q(t)dt = Q_0 t_0.$$

Interestingly, it does not depend on $\tau = R_0C$. Therefore, the acquired vertical velocity of the electron is

$$v_r = \eta r \frac{CV_0 d}{v} = \frac{2e\beta CV_0 dr}{mv}.$$

Following the logic similar to part C, we derive the focal length

$$f = -\frac{E}{eCV_0 d\beta}.$$

E.1 (1.7 pt)

$$f = -\frac{E}{eCV_0 d\beta}.$$

E.2 (0.3 points).

Comparing $f = -E/(eCV_0 d\beta)$ with $f = -E/(eqd\beta)$ from part C we immediataly obtain $q_{\mathrm{eff}} = CV_0$.

E.2 (0.3 pt)

$$q_{\mathrm{eff}} = CV_0.$$

Particles and Waves (10 points)

Part A. Quantum particle in a box (1.4 points)

A.1 (0.4 points)

The width of the potential well (L) should be equal to the half of the wavelength of the de Broglie standing wave $\lambda_{\mathrm{dB}} = h/p$, here h is the Planck's constant and p is the momentum of the particle. Thus $p = h/\lambda_{\mathrm{dB}} = h/(2L)$, and the minimal possible energy of the particle is

$$E_{\min} = \frac{p^2}{2m} = \frac{h^2}{8mL^2}.$$

A.1 (0.4 pt)

$$E_{\min} = \frac{h^2}{8mL^2}.$$

A.2 (0.6 points)

The potential well should fit an integer number of the de Broglie half-wavelengths: $L = \frac{1}{2}\lambda_{\mathrm{dB}}^{(n)} \cdot n$, $n = 1, 2, \ldots$. Therefore, particle's momentum, corresponding to the de Broglie wavelength $\lambda_{\mathrm{dB}}^{(n)}$ is

$$p_n = \frac{h}{\lambda_{\mathrm{dB}}^{(n)}} = \frac{hn}{2L},$$

and the corresponding energy is

$$E_n = \frac{p_n^2}{2m} = \frac{h^2 n^2}{8mL^2}, \qquad n = 1, 2, 3, \ldots. \tag{1}$$

A.2 (0.6 pt)

$$E_n = \frac{h^2 n^2}{8mL^2} = E_{\min} n^2.$$

A.3 (0.4 points)

The energy of the emitted photon, $E = hc/\lambda$ (here c is the speed of light and λ is the photon's wavelength) should be equal to the energy difference $\Delta E = E_2 - E_1$, therefore

$$\lambda_{21} = \frac{hc}{E_2 - E_1} = \frac{8mcL^2}{3h}.$$

A.3 (0.4 pt)

$\lambda_{21} = \dfrac{8mcL^2}{3h}$.

Part B. Optical properties of molecules (2.1 points)

B.1 (0.8 points)

Taking into account the Pauli exclusion principle, each energy level E_n is occupied by two electrons with spins oriented in the opposite directions. As a results, 10 electrons fill the lowest 5 states, and the absorption of the photon of the longest wavelength corresponds to the transition of one electron from the occupied E_5 to the unoccupied E_6 energy state:

$$\frac{hc}{\lambda} = E_6 - E_5,$$

where E_6 and E_5 can be found from Eq. 1, where m is replaced with the electron mass m_e. Hence we obtain:

$$\lambda = \frac{c \cdot 8m_e L^2}{h\,(6^2 - 5^2)} = \frac{10.5^2 \cdot 8}{11}\frac{m_e c l^2}{h} = \frac{882}{11}\frac{m_e c l^2}{h} \approx 647\,\text{nm}.$$

This result correspond precisely to the experimental value the peak position of the Cy5 absorption spectrum.

B.1 (0.8 pt)

Expression: $\lambda = \dfrac{882}{11}\dfrac{m_e c l^2}{h}$. Numerical value: $\lambda = 647\,\text{nm}$.

B.2 (0.4 points)

In the similar model for the Cy3 molecule, there are 8 electrons in the box of length $L = 8.5l$, thus photon's absorption corresponds to the $E_4 \longrightarrow E_5$ transition. Taking into account the result of question B1, we obtain

$$\lambda_{\text{Cy3}} = \frac{8.5^2 \cdot 8}{(5^2 - 4^2)}\frac{m_e c l^2}{h} \approx 518\,\text{nm},$$

i. e. the absorption spectrum of the Cy3 molecule is shifted by $\Delta\lambda \approx 129\,\text{nm}$ to the blue comparing to that of the Cy5 molecule. The experimental value is $\lambda_{\text{Cy3}}^{(\text{exp})} = 548\,\text{nm}$, so that our model catches general properties of these dye molecules rather well.

B.2 (0.4 pt)

Absorption spectrum of Cy3 is shifted to the (check): ☒ **bluer** ☐ redder

spectral region by $\Delta\lambda \approx 129\,\text{nm}$.

B.3 (0.7 points)

Let us assume

$$K = k\varepsilon_0^\alpha h^\beta \lambda^\gamma d^\delta. \tag{2}$$

The SI units of the relevant quantities are:

$$[\varepsilon_0] = \frac{\text{A}^2 \cdot \text{s}^4}{\text{kg} \cdot \text{m}^3}, \qquad [h] = \frac{\text{kg} \cdot \text{m}^2}{\text{s}}, \qquad [\lambda] = \text{m}, \qquad [d] = \text{A} \cdot \text{s} \cdot \text{m}, \qquad [K] = \text{s}^{-1}.$$

By plugging these expressions into Eq. 2 we obtain a simple system of linear equations for the unknown powers α, β, γ, and δ:

$$2\alpha + \delta = 0, \qquad -\alpha + \beta = 0, \qquad 4\alpha - \beta + \delta = -1, \qquad -3\alpha + 2\beta + \gamma + \delta = 0.$$

By solving this system we get:

$$\alpha = \beta = -1, \qquad \gamma = -3, \qquad \delta = 2,$$

so that the rate of spontaneous emission is

$$K = \frac{16\pi^3}{3} \frac{d^2}{\varepsilon_0 h \lambda^3}. \tag{3}$$

B.3 (0.7 pt)

$$K = \frac{16\pi^3}{3} \frac{d^2}{\varepsilon_0 h \lambda^3}.$$

B.4 (0.2 points)

By using the result of question B.2 and expressing transition dipole moment as $d = 2.4\,el$, we obtain from Eq. 3:

$$\tau_{\text{Cy5}} = \frac{3}{16\pi^3} \frac{\varepsilon_0 h}{2.4^2 l^2 e^2} \lambda^3 \approx 3.3\,\text{ns}.$$

B.4 (0.2 pt)

Numerical value: $\tau_{\text{Cy5}} \approx 3.3\,\text{ns}$.

Part C. Bose-Einstein condensation (1.5 points)

C.1 (0.4 points)

At temperature T, the average kinetic energy of translational motion is $\frac{3}{2}k_B T$. Equating this result to $p^2/(2m)$, we obtain typical momentum $p = \sqrt{3mk_B T}$ and the de Broglie wavelength

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{\sqrt{3mk_B T}}.$$

C.1 (0.4 pt)

$p = \sqrt{3mk_B T}$. $\lambda_{dB} = \dfrac{h}{\sqrt{3mk_B T}}$.

C.2 (0.5 points)

The volume per particle V/N is a good estimate for ℓ^3. We obtain $\ell = n^{-1/3}$, with $n = N/V$ and equate $\ell = \lambda_{dB}$ to express $T_c = h^2 n^{2/3}/(3mk_B)$.

C.2 (0.5 pt)

$\ell = n^{-1/3}$. $T_c = \dfrac{h^2 n^{2/3}}{3mk_B}$.

C.3 (0.6 points)

Using the answer to the previous question, we express $n_c = (3mk_B T_c)^{3/2}/h^3$. Equation of state for the ideal gas gives $n_0 = p/(k_B T)$. Numerical estimations yield $n_c \approx 1.59 \cdot 10^{18}\ \mathrm{m}^{-3}$ and $n_0/n_c \approx 1.5 \cdot 10^7$.

C.3 (0.6 pt)

Expression: $n_c = \dfrac{(3 \cdot 87\, m_{\mathrm{amu}} k_B T_c)^{3/2}}{h^3}$. Numerical value: $n_c \approx 1.59 \cdot 10^{18}\ \mathrm{m}^{-3}$.

Expression: $n_0 = p/(k_B T)$. Numerical value: $n_0/n_c \approx 1.5 \cdot 10^7$.

Part D. Three-beam optical lattices (5.0 points)

D.1 (1.4 points)

We sum the three electric fields (z components)

$$E(\vec{r},t) = E_0 \sum_{i=1}^{3} \cos\left(\vec{k}_i \cdot \vec{r} - \omega t\right), \tag{4}$$

and square the result

$$\begin{aligned} E^2(\vec{r},t) &= E_0^2 \sum_{i=1}^{3}\sum_{j=1}^{3} \cos\left(\vec{k}_i \cdot \vec{r} - \omega t\right)\cos\left(\vec{k}_j \cdot \vec{r} - \omega t\right) \\ &= \frac{E_0^2}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left\{\cos\left[\left(\vec{k}_i - \vec{k}_j\right)\cdot\vec{r}\right] + \cos\left[\left(\vec{k}_i + \vec{k}_j\right)\cdot\vec{r} - 2\omega t\right]\right\}. \end{aligned} \tag{5}$$

Time averaging gives

$$\langle E^2(\vec{r},t)\rangle = \frac{E_0^2}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\cos\left[\left(\vec{k}_i - \vec{k}_j\right)\cdot\vec{r}\right], \tag{6}$$

we analyse the 9 terms and simplify to

$$\langle E^2(\vec{r},t)\rangle = E_0^2\left(\frac{3}{2} + \sum_{j=1}^{3}\cos\vec{b}_j\cdot\vec{r}\right). \tag{7}$$

Here $\vec{b}_{1,2,3} = (\vec{k}_2 - \vec{k}_3), (\vec{k}_3 - \vec{k}_1), (\vec{k}_1 - \vec{k}_2)$ or in terms of the Levi-Civita symbol, $\vec{b}_k = \varepsilon_{ijk}(\vec{k}_i - \vec{k}_j)$. Incidentally, they are known as the reciprocal lattice vectors.

D.1 (1.4 pt)

$$V(\vec{r}) = -\alpha E_0^2\left(\frac{3}{2} + \sum_{j=1}^{3}\cos\vec{b}_j\cdot\vec{r}\right).$$

$\vec{b}_1 = \vec{k}_2 - \vec{k}_3,\qquad \vec{b}_2 = \vec{k}_3 - \vec{k}_1,\qquad \vec{b}_3 = \vec{k}_1 - \vec{k}_2.$

D.2 (0.5 points)

D.2 (0.5 pt)

Argument: Observe that rotation by 60° maps the three vectors $\vec{b}_{1,2,3}$ into the relabelled triplet of $-\vec{b}$'s.

D.3 (1.2 points)

We find

$$V(x,y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos\left(ky\sqrt{3}\right) + \cos\left(\frac{3kx}{2} + \frac{ky\sqrt{3}}{2}\right) + \cos\left(\frac{3kx}{2} - \frac{ky\sqrt{3}}{2}\right) \right\}, \tag{8}$$

and deduce

$$V_X(x) = -\alpha E_0^2 \left\{ \frac{5}{2} + 2\cos\frac{3kx}{2} \right\}. \tag{9}$$

The potential has a simple cosine form, and the origin in an obvious minimum. Its replica appear at multiples of $\Delta x = 4\pi/(3k)$. In the midpoint between any two minima, e.g. at $x = \Delta x/2 = 2\pi/(3k)$, the function $V_X(x)$ has its maxima.

Concerning the behaviour along the y axis, we have

$$V_Y(y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos 2\varphi + 2\cos\varphi \right\}, \qquad \varphi = \sqrt{3}ky/2. \tag{10}$$

Looking for the extrema, we find the equation

$$\sin 2\varphi + \sin\varphi = 0. \tag{11}$$

- $\varphi = 0$ is the ‘deep’ minimum – the lattice site;
- $\varphi = \pi$ is the ‘shallow’ minimum (later shown to be a saddle point of $V(x,y)$);
- $\varphi = 2\pi/3$ and $\varphi = 4\pi/3$ are maxima.

D.3 (1.2 pt)

$V_X(x) = -\alpha E_0^2 \left\{ \frac{3}{2} + 2\cos\frac{3kx}{2} \right\}.$

$V_Y(y) = -\alpha E_0^2 \left\{ \frac{3}{2} + \cos 2\varphi + 2\cos\varphi \right\}, \qquad$ here $\varphi = \sqrt{3}ky/2.$

Minimum (-a) of $V_X(x)$: $x = 0$.
Maximum (-a) of $V_X(x)$: $x = \frac{2\pi}{3k}$.

Minimum (-a) of $V_Y(y)$: $y = 0$ (‘deep’) and $y = \frac{2\pi}{\sqrt{3}k}$ (‘shallow’).

Maximum (-a) of $V_Y(y)$: $y = \frac{4\pi}{3\sqrt{3}k}$ and $y = \frac{8\pi}{3\sqrt{3}k}$.

D.4 (0.8 points)

We review the minima found in the previous question and eliminate the saddle point at $(0, 2\pi/3\sqrt{3}k)$. The actual minima of the 2D potential landscape $V(x, y)$ are:

- $(0, 0)$ – at the origin;
- $(4\pi/(3k), 0)$ – nearest to the origin in the positive direction along the x axis. On the grounds of symmetry we argue that there are six equivalent nearest minima in the directions $0°$, $\pm60°$, $\pm120°$, and $180°$ with respect to the x axis.

Distance between nearest minima (the lattice constant) $a = 4\pi/(3k)$. Given that the laser wavelength is $\lambda_{\text{las}} = 2\pi/k$, we have $a = \Delta x = 2\lambda_{\text{las}}/3$.

D.4 (0.8 pt)

Ratio of the lattice constant to the laser wavelength: $a/\lambda_{\text{las}} = \frac{2}{3}$

Positions of all equivalent minima nearest to the origin: **in the directions 0°, ±60°, ±120°, and 180° with respect to the x axis**.

D.5 (1.1 points)

The atom's core electrons (all but the one promoted to to a state with a high principal quantum number n) shield the electric field of the nucleus so that the effective potential for the outer electron resembles that of a hydrogen atom. The attractive force acting on that electron, $F = e^2/(4\pi\varepsilon_0 r^2)$, gives rise to its centripetal acceleration $a = v^2/r$. Equating $F = m_e a$ and using the expression for the angular momentum $m_e v r = n\hbar$ to eliminate the velocity, we find the quantum number n corresponding to the orbit with the radius $r = \lambda_{\text{las}}$:

$$n = \frac{e}{\hbar}\sqrt{\frac{m_e \lambda}{4\pi\varepsilon_0}} \approx 85. \tag{12}$$

D.5 (1.1 pt)

Expression: $n = \frac{e}{\hbar}\sqrt{\frac{m_e \lambda}{4\pi\varepsilon_0}}$. Numerical value: $n \approx 85$.