Permanent magnets (10 points)

Strong permanent magnets are made from NdFeB alloy which obeys a very wide hysteresis loop so that the magnetization J can be assumed to be constant over a wide range of applications; in what follows, we assume that $J \equiv 1.5\,\mathrm{T}/\mu_0$, where $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$, and the magnetization of all the permanent magnets is homogeneous. *Magnetization is defined as the volume density of the magnetic dipole moment of the matter.*

Hint 1. The following equality might be useful:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

*Hint 2.*The magnetic field created by a spherical magnet is identical to that of a point dipole. The magnetic fields created by magnets of other shapes become equivalent to a point dipole fields only at distances much larger than their diameter.

Hint 3. Electric and magnetic fields of electric and magnetic point dipoles as functions of coordinates and of the corresponding dipole moment are similar, i.e. one can be obtained from the other by multiplying it by a constant factor.

Hint 4. The induced field due to a boundary condition can always be replaced by some configuration of field sources outside the given boundaries.

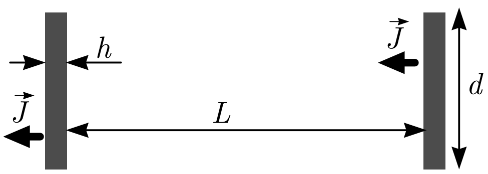
Part A. Interaction of magnets (4.5 points)

When the distance to a magnet is much larger than its size, the magnetic field created by it can be approximated with the magnetic field of its dipole moment $\vec{m}$,

$$\vec{B} = \frac{\mu_0}{4\pi r^3}(2\vec{m}_{\parallel} - \vec{m}_{\perp}).$$

Here $r = |\vec{r}|$, and we have decomposed the dipole moment into components parallel and perpendicular to the radius vector $\vec{r}$ drawn from the dipole to the observation point, $\vec{m} = \vec{m}_{\perp} + \vec{m}_{\parallel}$.

A.1	Find the magnitude of the interaction force between two coaxially placed cylindrical magnets of diameter $d = 20\,\mathrm{mm}$ and thickness $h = 2\,\mathrm{mm}$, magnetized parallel to their axis, if the distance between the centres of the magnets is $L = 20\,\mathrm{cm}$. You may assume that $L \gg d, h$.	0.6pt

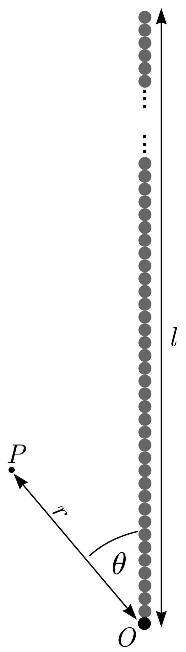


A.2	At distances much larger than $\frac{h}{2}$, the field created by the magnet from task A.1 is the same as that created by a circular current I. Find I.	0.4pt

A.3	Find the interaction force between the magnets for the setup of task A.1 if instead $L = 5$ mm. You may assume that $d \gg L \gg h$.	1.0pt

A.4	Identical spherical magnets of diameter $\delta = 5$ mm, bound together by magnetic attraction, form a chain. What is the maximal length l for such a chain which does not break under its own weight when hanging beneath the topmost magnet? The density of NdFeB magnets $\rho = 7500\,\mathrm{kg/m^3}$.	1.0pt

A.5	Consider the chain from part A.4. Obtain an expression for the magnitude of the magnetic B-field at such a point P which is at distance r from one of the chain's endpoint O, and the angle between the chain and the line OP is θ (cf. figure below), assuming that $l \gg r$ and $r \sin\theta \gg \delta$	1.5pt



Part B. Interaction with ferromagnets (3.5 points)

Now we assume that in addition to the permanent magnets we have also plates made from a ferromagnetic material, similar to what is used in transformer cores. In the situations we're concerned with, it can be considered to have a constant but very large relative permeability $\mu_r \sim 10^5$.

Hint 5. Large permeability means that magnetic field lines near the outside surface of an object made of the material are nearly perpendicular to the surface. This is similar to the behavior of electric field lines near the outside surface of a conductor.

B.1	A spherical magnet from part A.4 is at a distance $s = \delta$ from a thick infinite ferromagnetic plate (see the answer sheet). The magnetization of the sphere is oriented perpendicular to the plate. Sketch the field lines in the cross-section shown in the answer sheet. In that figure, three points (denoted as 1, 2, and 3) are marked; you need to show field lines passing through each of these points in their full length, i.e. as much as fits into the figure.	1.0pt

B.2	Now the spherical magnet is brought into direct contact with the plate. Which direction is taken by the magnetization vector of the spherical magnet at a stable equilibrium and what is the normal force between the plate and the magnet? Mark the correct direction(s) with a tick in the corresponding box in the answer sheet. Incorrect ticks will reduce your score.	1.0pt

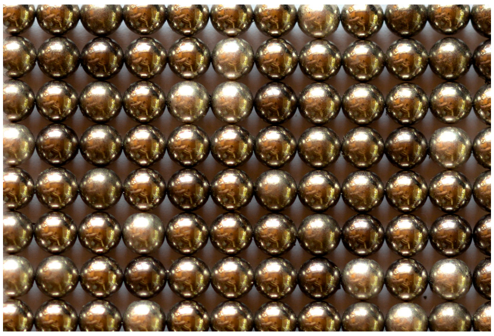
B.3	Now a magnet from part A.1 is placed between two thick circular ferromagnetic plates of diameter $D = 2d$ so that the flat faces of the magnet are pressed against the plates and all three discs are coaxial. Find the magnetic force F acting on each plate. *Hint:* You may neglect the magnetic field **both** outside the ferromagnetic plates and outside the gap between them.	1.5pt

Part C. (Anti)ferromagnetic order (2 points)

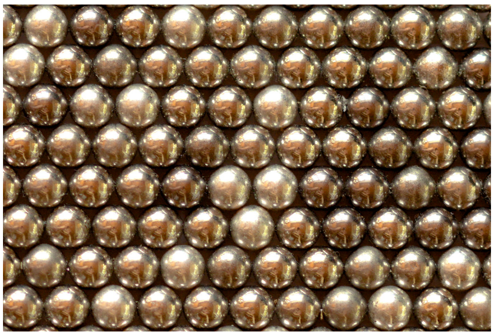
The magnetic properties of materials are due to the magnetic dipole moments of electrons and atomic nuclei. If the dipole moments orient themselves parallel to each other, the field created by them is magnified — these are ferromagnetic materials. On the other hand, if for each dipole moment there is another antiparallel dipole moment nearby, the fields cancel out — these are anti-ferromagnetic materials.

In what follows, we consider a very large number of spherical magnets of part A.4, arranged at the nodes of a two-dimensional lattice; see **real photos of stable equilibrium configurations** below. Assume that all the magnetization vectors lie in the plane of the figure. Consider in your calculations only nearest-neighbour interactions (on the figure of C.1, each magnet has four nearest neighbours, and on the figure of C.2 — six).

C.1	Show the magnetization directions of the magnets in the figure below. You are not required to prove that the configuration you suggested is the only possibility. You still need to justify that the configuration you suggested is indeed stable. Find the energy needed to pull one magnet out of this lattice from somewhere in the middle of the lattice, assuming the other magnets are kept stationary. Does this configuration correspond to the order of ferromagnetic or antiferromagnetic materials?	0.8pt



C.2	Answer the same questions as in task C.1 for the configuration shown in the figure below.	1.2pt



James Webb Space Telescope (12 points)

This is a question on the physics of the James Webb Space Telescope. Light from a star strikes the primary mirror, with an area of $A_{\text{mirror}} = 25\,\text{m}^2$, and reflects off of a secondary mirror. The focal length of the system is $f = 130\,\text{m}$. The light is focused into the ISIM (Integrated Science Instrument Module), which contains the CCD (charged-coupled device) cameras.

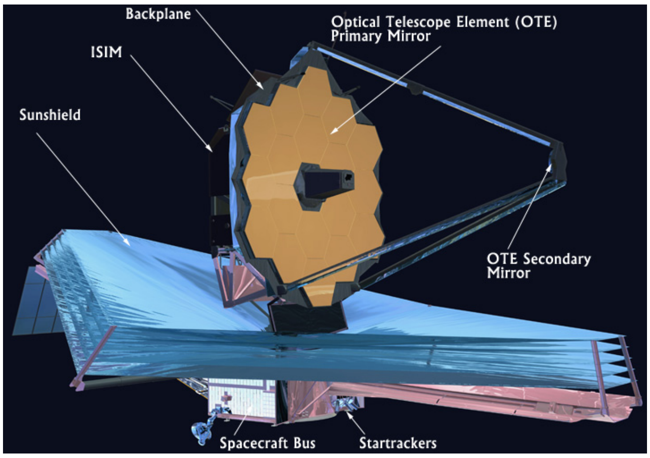


Image Credit: NASA

Part A. Imaging a Star (1.8 points)

The nearest Red Giant is 89 light-years distant, has a temperature of $T_{\text{star}} = 3600\,\text{K}$, and a diameter of $d_o = 1.7 \times 10^{11}\,\text{m}$.

A.1	Calculate the diameter of a focused image of the star on the CCD camera imaging surface.	0.4pt

A.2	Estimate the diameter of a diffraction central maximum on the CCD camera imaging surface. Assume a wavelength of $\lambda = 800\,\text{nm}$, which is the strongest intensity wavelength from the red giant star.	0.4pt

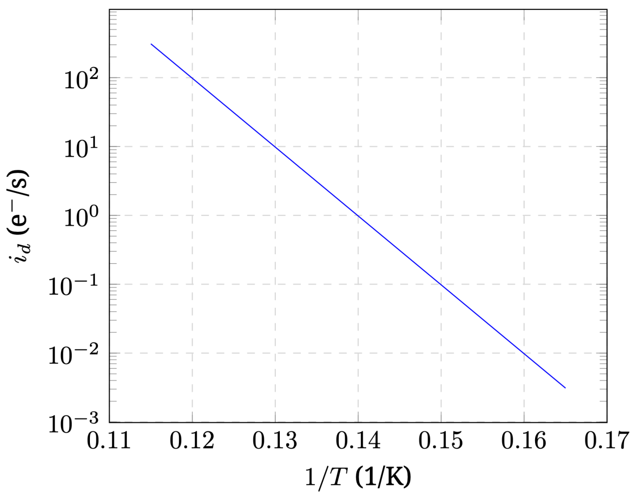
A.3	If the CCD is not cooled and can lose heat only by radiating from the top of the imaging surface, what would be the equilibrium temperature of the CCD at the location of the image of the red giant star? Assume the CCD surface is a blackbody. Provide a formula and a numerical estimate.	1.0pt

Part B. Counting Photons (1.8 points)

The absorption of a photon by the CCD camera leads to the emission of an electron within the apparatus. This occurs only if the photon has sufficient energy to excite an electron across an energy gap ΔE_g. Assume that every photon with sufficient energy succeeds. There is also leakage of electrons across the gap caused by the temperature of the CCD camera; this is the dark current i_d and is measured in the number of electrons per second. It is a function of temperature according to

$$i_d = i_0 e^{-|\Delta E_g|/6k_B T}. \tag{1}$$

where i_0 is a constant.



The graph shows how dark current varies with temperature. The units for dark current, e$^-$/s should be thought of as counting a number of electrons per second.

B.1	From the dark current graph, provide an order of magnitude estimate for the temperature of a distant source of thermal photons that would just be capable of exciting an electron on the pixel.	0.4pt

The electrons are collected in a capacitor, and after an exposure time τ, the electrons are counted. There are three main sources of uncertainty in the process: a fixed uncertainty in the counting process called read out noise; a Poisson distribution error associated with the dark current, and a Poisson distribution error associated with the detected incoming photons. Poisson distribution errors are equal to the square root of the number of counts associated with a process. The measured photon count is equal to the number of electrons in the capacitor, minus the number of electrons associated with the dark current.

B.2	Write an expression for the total count uncertainty σ_t, if there is a readout noise σ_r, a dark current i_d, an incoming photon rate p, and an exposure time τ.	0.4pt

For remaining questions in this part assume the exposure time is $\tau = 10^4$ s and the read out noise is a fixed $\sigma_r = 14$.

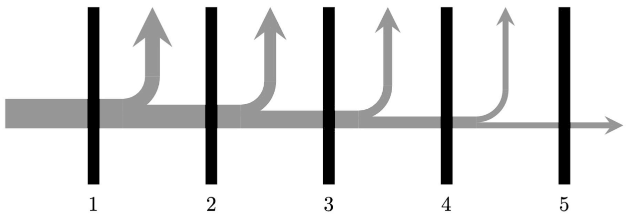
B.3	Assume an operating temperature of $T_\text{p} = 7.5$ K. Calculate the minimum photon rate p so that the photon count is ten times the count uncertainty.	0.5pt

B.4	Assuming all photons are just capable of exciting an electron across the band gap, what is the intensity of the source of photons found in B.3 on the primary mirror? Express your answer in W/m^2	0.5pt

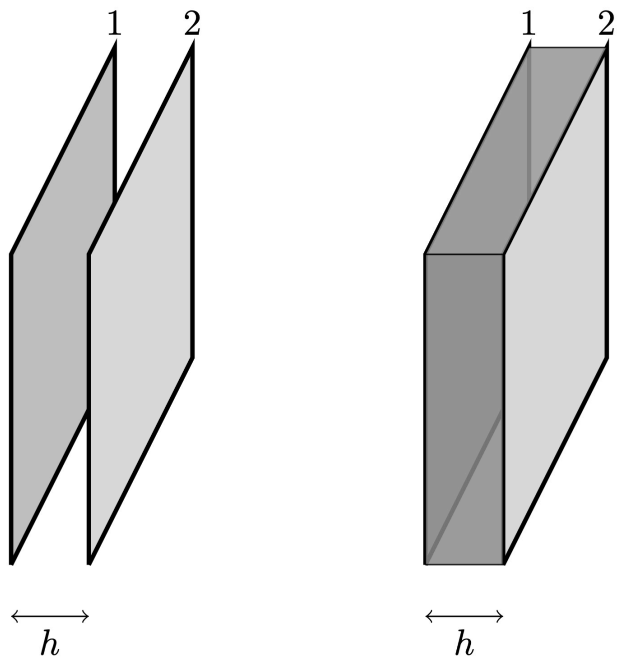
Part C. Passive Cooling (4.4 points)

An infrared CCD camera must be kept at a low temperature. The first tool is a shield to protect from the sun's radiation.

The sun-shield consists of five separated reflective layers in thin sheets (black); radiant energy (gray) from the sun is incident on the first sheet on left, and some energy escapes between every pair of sheets.



Schematic of energy flow: the vertical lines (black) are the sheets, the flow of energy (gray) is from the left to the right, however, between sheets, some energy flows up and out into space.



On the left is a simple model of two adjacent sheets 1 and 2 separated by a distance h. The sheets are not connected, and the perimeter is open to space. Assume the sheets are parallel. Thermal radiation can be exchanged between the sheets, and thermal radiation can escape through the perimeter gap. On the right, the perimeter gap has been shaded to help visualize.

Assume the following simplifications:

- Sheets are square, each with area $A_{\text{sheet}} = 200\,\text{m}^2$.
- Sheets are parallel and separated by $h = 25\,\text{cm}$ along the perimeter.
- Sheets have constant emissivity $\epsilon \ll 1$. Assume that all reflections off of sheet surfaces are diffuse.
- Sheets are thin with temperature on the front and back surfaces equal and uniform.
- The fraction of radiant flux emitted by a sheet that is absorbed by the adjacent sheet is $\alpha \leq 1$. This means that if sheet 1 in the figure above emits an amount of heat Q_1toward sheet 2 then sheet 2 will absorb an amount $\alpha Q_1$from sheet 1.
- The amount of radiant flux ejected out of the perimeter gap between two sheets is approximated as βQ_{12} where αQ_{12} is the net flux between the two sheets. The fraction $\beta < 1$. This is equivalent to saying that the heat loss to space between two sheets is proportional to the net heat exchange between the sheets. This is a rough approximation for this problem.
- Background temperature of space is negligible.

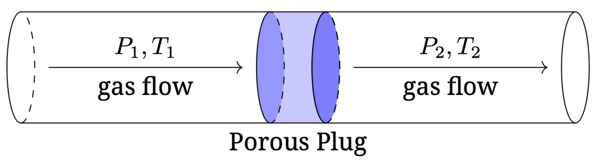
C.1	Derive expressions for the equilibrium temperatures of the first sheet and fifth sheet in terms of the incident solar radiation intensity I_0, the constants α and β, and any necessary physical constants. To simplify your expression, you may define additional constants in terms of α and β, etc.	2.4pt

C.2	Derive numerical estimates for α and β from the information about the sheet geometry assuming an emissivity $\epsilon = 0.05$. You are encouraged to consider the rectangular box model of the sheets above, where the perimeter area effectively acts as a perfect absorber of radiant energy.	1.6pt

C.3	Numerically determine the temperatures of sheet 1 and sheet 5. The solar intensity is $I_0 = 1360\,\text{W/m}^2$.	0.4pt

Part D. Cryo-cooler (4 points)

The last stage of the cooling system directly cools the CCD camera. A closed cycle refrigeration system has a supply pipe line feeding helium gas at constant pressure P_1moving through a sponge like porous plug into a pipe with constant pressure P_2. The pipe carries the gas to cool the CCD. The helium gas then passes through a pump before returning to the supply line.



Helium gas supplied on the left at well defined pressure P_1 and temperature T_1 is forced through the plug to well defined pressure P_2 and temperature T_2, where it is carried away on the right.

As the gas moves through the porous plug, viscous friction with the narrow walls of the channels in the sponge becomes an important effect; however, no heat is transferred to or from the gas during the process. The bulk speed of the gas in region 2 is only marginally greater than the bulk speed in region 1.

Helium is not an ideal gas, but does remain in a gaseous state throughout this process.

D.1	Consider a mole of gas that passes from left to right through the plug. Complete the table in your answer sheet by writing '>' or '<' to identify the quantity that must be greater, '=' to identify quantities that must be equal, or '?' if it is not possible to know which is greater or equal without more information.	1.0pt

D.2	Identify a conserved quantity constructed from U (internal energy), P (pressure), and V (volume) as a mole of gas moves through the plug; show work on how you derived this conserved quantity.	0.6pt

Your answer sheets have graphs of internal energy per mass against volume per mass for helium with isotherms and lines of constant entropy.

D.3	Assuming that $V_2 = 0.100\,\mathrm{m^3/kg}$ and $T_2 = 7.5\,\mathrm{K}$, use the graph to find a numerical value for the conserved quantity that you found in Part D.2. Show the construction on the graph!	1.4pt

D.4	Find the maximum possible temperature for T_1. Show the construction on the graph!	0.8pt

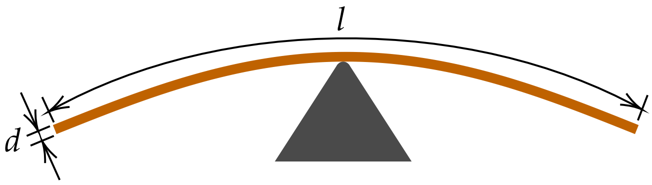
D.5	Assuming your value for the maximum T_1 found in D.4, find a numerical value for P_1.	0.2pt

Scaling laws (8 points)

Scaling laws describe the functional relationship between two physical quantities that scale with each other over a significant interval. This functional relationship can be a power law, but there are other possibilities, too. Oftentimes, exact expressions are beyond reach, but scaling laws can still be derived.

Part A. Spaghetto (2.0 points)

A.1 A spaghetto straw of diameter d is being balanced horizontally from its middle. If $d = 1\,\text{mm}$, the straw breaks under its own weight once its length reaches $l = 50\,\text{cm}$. What is the maximum length l' of the straw of diameter $d' = 1\,\text{cm}$ before it breaks under its own weight? 2.0pt



Part B. Sand castle (2.0 points)

B.1 The average grain volume of coarse-grained sand is 10 times as large as that of fine-grained sand. Wet fine-grained sand and wet coarse-grained sand have both optimal water content (i.e. assuming the maximal strength of the constructions from it) and are used to build two cylinders of exactly the same shape and size. The strength of each cylinder is tested by pressing it between two parallel plates. The cylinder made of coarse-grained sand gets destroyed once the force applied to press the plates reaches $F_\text{c} = 10\,\text{N}$. How large is the force F_f needed to destroy the cylinder made of fine-grained sand? You may ignore the effects of gravity. 2.0pt

Part C. Interstellar travel (2.0 points)

C.1	The spaceship of an interstellar expedition travels at a constant magnitude of the proper acceleration $g = 10\,\mathrm{m/s^2}$, i.e., this is the acceleration of the spaceship in the inertial frame of reference where it is instantaneously at rest. The passengers must be able to return to Earth within their remaining expected lifetime of 50 years. The maximum distance from Earth reached by the spaceship is d. If the acceleration is increased to $g' = 15\,\mathrm{m/s^2}$, the spaceship can reach a farther distance d'. What is the ratio d'/d? *Hint 1.* You may wish to use the relativistic velocity addition formula, however, there are also other approaches. *Hint 2.* You may need to deal with hyperbolic functions defined as follows: $\cosh x = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$, $\sinh x = \frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})$, $\tanh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}$. *Hint 3.* Depending on your approach, you may need one or more of these integrals: $\int \frac{\mathrm{d}x}{1-x^2} = \operatorname{atanh} x + C$, $\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \operatorname{asinh} x + C$, $\int \sinh x \mathrm{d}x = \cosh x + C$, where $\operatorname{asinh} x$ and $\operatorname{atanh} x$ are the inverse functions of the respective hyperbolic functions.	2.0pt

Part D. That sinking feeling (2.0 points)

D.1	A solid wooden ball of radius r_0 is floating in the water. Ignoring frictional effects, the frequency of small oscillations would be ω_0, but because of viscous friction, after being displaced vertically, the frequency of decaying oscillations is actually $0.99\,\omega_0$. What is the minimum radius r_{min} of a wooden ball floating in water that undergos small oscillations when displaced? *Hint: t*he viscous drag force acting on a given body is proportional to its speed relative to the bulk of the fluid, and to the viscosity η of the fluid it is moving in. The unit of the viscosity is $\mathrm{kg/(m \cdot s)}$.	2.0pt