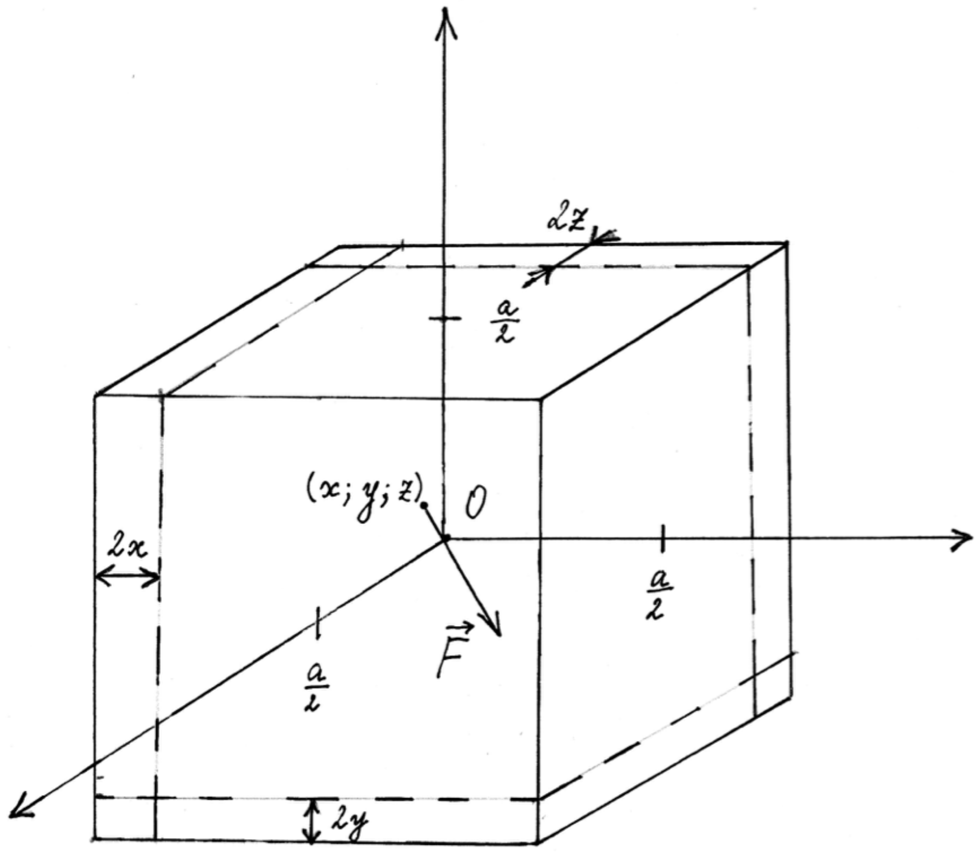
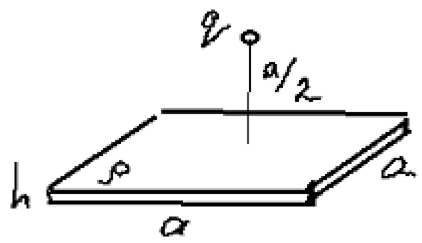
A1 2.50 A narrow straight channel passes through the center of a fixed cube with a side a. The cube is uniformly charged, the charge density is ρ. The distance from the cube center to the point of intersection of the channel and a face is L. In the channel there is a particle of a mass m and a charge q. Find the period of small oscillations of the particle near the center. The gravitational interaction of the particle and the cube can be neglected. The cube and the particle are oppositely charged.

We will use a coordinate system with axes parallel to the cube's edges, with the origin set at the cube's center. Assuming the particle is at coordinates (x, y, z) $x \ll a$, $y \ll a$, $z \ll a$, we will find the force $\vec{F}$ acting on the particle, by splitting the cube $a \times a \times a$ into a rectangular cuboid $(a-2x) \times (a-2y) \times (a-2z)$ and three square plates of thickness $2x$, $2y$ and $2z$.



The particle is in the center of the cuboid, so there is no force from the cuboid.
Let us find the force between a particle with a charge q and a uniformly charged square plate of small thickness h and edge length a. The plate's charge density is ρ, the particle is placed above the center of the plate at distance $a/2$.



Due to symmetry and Gauss's law, the flux of the particle's electric field through the plate is

$$\Phi = \frac{q}{6\varepsilon_0}.$$

Hence, the force

$$F = \sigma\Phi = \frac{q\rho h}{6\varepsilon_0},$$

where $\sigma = \rho h$ is plate's surface charge density.
Three square plates act on the particle with forces $\vec{F}_1 = \frac{q\rho x}{3\varepsilon_0}\hat{x}$, $\vec{F}_2 = \frac{q\rho y}{3\varepsilon_0}\hat{y}$, and $\vec{F}_3 = \frac{q\rho z}{3\varepsilon_0}\hat{z}$. Net force $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \frac{q\rho}{3\varepsilon_0}\vec{r}$, where $\vec{r}$ is the position vector of the particle.

The particle's equation of motion

$$m\ddot{\vec{r}} = \frac{q\rho}{3\varepsilon_0}\vec{r}$$

is an equation of simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{3m\varepsilon_0}{q(-\rho)}}.$$

B1 3.00 A current I flows through a loop made of a weightless flexible wire. The loop upper point is attached to the ceiling and a weight is suspended to its lowest point. The half length of the loop is L. The loop is placed in a vertical magnetic field B. The system has reached a stable equilibrium in which the point of suspension at the ceiling and the point of weight suspension are not on the same vertical. Find the wire tension T and the weight P if the distance from the ceiling to the lowest point of the loop is H.

First approach

The Ampere's force acting on a short piece of wire of length $\vec{\mathrm{d}l}$ is

$$\vec{F}_{amp} = I\vec{\mathrm{d}l} \times \vec{B}.$$

The net force acting on this piece is

$$\vec{F} = \vec{T}_1 + \vec{T}_2 + I\vec{\mathrm{d}l} \times \vec{B} = 0. \qquad (1)$$

Here $\vec{T}_1$ and $\vec{T}_2$ are tension force of the wire acting on the two end points of the piece.

Projected on the $\vec{\mathrm{d}l}$, this equation reads:

$$T_1 = T_2.$$

Therefore, the tension is constant along the wire.

Let $\vec{n}(l)$ be the tangent vector of the wire at the distance l from the point of suspension. The equation (1) reads

$$\vec{T}_1 + \vec{T}_2 + I\vec{\mathrm{d}l} \times \vec{B} = \mathrm{d}l(T\frac{\mathrm{d}\vec{n}(l)}{\mathrm{d}l} + I\vec{n}(l) \times \vec{B}) = 0,$$

$$\frac{\mathrm{d}\vec{n}(l)}{\mathrm{d}l} = -\frac{I}{T}\vec{n}(l) \times \vec{B}.$$

This equation implies that

$$\vec{B} \cdot \frac{\mathrm{d}\vec{n}(l)}{\mathrm{d}l} = 0 \implies \vec{B} \cdot n(l) = \mathrm{const}.$$

The tangent vector of the wire $\vec{n}$ is at constant angle to the magnetic field. In the horizontal plane the tangent vector is rotating at a constant speed. Thus, each side of the loop has the cylindrical helix shape, i. e. it is winding around some cylinder and making the constant angle α to the magnetic field. To find the radius of this cylinder, let's write the equation in the projection on the horizontal plane:

$$\frac{\mathrm{d}\vec{n}_{xy}}{\mathrm{d}l_{xy}}\sin(\alpha) = -\frac{I}{T}\vec{n}_{xy} \times \vec{B}.$$

$$R = \frac{T}{IB}\sin(\alpha). \qquad (2)$$

At the point of the weight suspension we have the balance between three forces: two forces of tension(from each side of the loop) and the gravitational force of the weight, which is vertical. These three forces should be in the same vertical plane. Therefore, both sides of the loop are winding around the same cylinder. Each half of the loop should make a half-turn around this cylinder.

Thus, the length of the half of the loop is

$$L = \sqrt{H^2 + (\pi R)^2}.$$

Therefore,

$$R = \frac{1}{\pi}\sqrt{L^2 - H^2} = \frac{1}{\pi}L\sin(\alpha). \qquad (3)$$

The tension force of the wire can be found from equations (2) and (3)

$$T = \frac{IBR}{\sin(\alpha)} = \frac{IBL}{\pi}.$$

The suspended weight is

$$P = 2T\cos(\alpha) = 2T\frac{H}{L}.$$

$$P = \frac{2IBH}{\pi}.$$

Second approach

Stable equilibrium corresponds to a minimum of potential energy. The potential energy is a sum of the gravitational energy of the weight and of the energy of the loop in the magnetic field. Let S be an area of the loop projected on the horizontal plane.

$$E_{\mathrm{p}} = -PH - ISB.$$

E_{p} depends on two variable parameters: H and S. Note that we can change the form of the projection (not changing its length) of the loop onto horizontal plane without changing H. Thus, the minimum of the potential energy corresponds to the maximum area of the projection with fixed length — a circle.

Now, if the projection is fixed, the height H is maximal if the wire makes a constant angle with the vertical axis. Thus, the wire has a shape of cylindrical helix — each side is winding a cylinder, while making constant angle α with vertical axis.

Let R be the radius of the cylinder.
Then

$$S = \pi R^2 = \frac{1}{\pi}(L^2 - H^2).$$

$$E_{\mathrm{p}} = -PH - IB\frac{1}{\pi}(L^2 - H^2).$$

The condition for the minimum of the potential energy is

$$\frac{\partial E}{\partial H} = 0.$$

$$P = \frac{2IBH}{\pi}.$$

The force balance equation at the point of weight suspension:

$$2\frac{H}{L}T = P.$$

$$T = \frac{LIB}{\pi}.$$

C1 4.50 A weightless rod of a length $2R$ is placed perpendicular to a uniform magnetic field $\vec{B}$. Two identical small balls of mass m and charge q each are attached at the rod ends. Let us direct z-axis along the magnetic field and place the origin at the rod center. The balls are given the same initial velocity v but in opposite directions so that one of the velocities is precisely in the z-direction. What are the maximum coordinates $z_{\max}$ of the balls? Express your answer in terms of q, B, m, v, and R. Find the magnitude of the ball accelerations at this moment and express your answer in terms of q, B, m, v, R, and $z_{\max}$.

The equations of motion of the centre of mass of balls is:

$$2m\dot{\vec{v}}_C = q\left[(\vec{v}_1+\vec{v}_2)\times\vec{B}\right] = q\left[\vec{v}_C\times\vec{B}\right],$$

where $\vec{v}_C$ is the velocity of the centre of mass, $\vec{v}_1$ and $\vec{v}_2$ are velocities of balls. Notice that $v_C = 0$ at the start of motion then v_C remains equal to 0 always.

The equation for the rotation of the ball about the centre of mass is:

$$\frac{d\vec{L}}{dt} = q\left[\vec{R}\times\left[\vec{v}\times\vec{B}\right]\right].$$

Now let's use the formula of a triple vector product:

$$\left[\vec{R}\times\left[\vec{v}\times\vec{B}\right]\right] = \vec{v}\left(\vec{R}\cdot\vec{B}\right) - \vec{B}\left(\vec{R}\cdot\vec{v}\right).$$

So balls move around the sphere with the center in the middle of rod. It means that $\vec{R}\perp\vec{v}$ then $\vec{R}\cdot\vec{v}=0$. Hence

$$\frac{d\vec{L}}{dt} = q\vec{v}\left(\vec{R}\cdot\vec{B}\right)$$

Firstly, $\vec{B}$ directs along z-axis, then $\left(\vec{R}\cdot\vec{B}\right) = Bz$, where z is a coordinat of the ball. Secondly, let look at projection of vector equation onto z-axis:

$$\frac{dL_z}{dt} = qv_zBz = qBz\frac{dz}{dt}.$$

Multiplying to dt and integrating gives us this equation:

$$L_z = \frac{qBz^2}{2} + C,$$

where $C=0$ because $L_z = 0$ at the start of the motion when $z=0$.

When z is maximum $v_z = \dot{z} = 0$. It means, that $L = L_z = mur$, where r is a distance between the ball and z-axis, u is the velocity of the ball at this moment. Recall that the ball is moving around the sphere then $r^2+z^2=R^2$.

The Lorenz force and the contact ball-rod force don't do the work. It means the kinetic energy of the system is constant. Together with $\vec{v}_C = 0$, it gives that velocities of the balls don't change i.e. $u=v$.

To summarize, $L_z = mv\sqrt{R^2 - z_{\max}^2}$.

Let's combine the results of the two last blocks as a biquadratic equation for $z_{\max}$:

$$\left(\frac{qB}{2mv}\right)^2 z_{\max}^4 + z_{\max}^2 - R^2 = 0.$$

Solution is:

$$z_{\max} = \frac{\sqrt{2}mv}{qB}\sqrt{\sqrt{1+\left(\frac{qBR}{mv}\right)^2} - 1}$$

This result can be obtained in a faster way by those who are familiar with the generalized momentum of a charged particle in a magnetic field,

$$\vec{\mathcal{P}} = m\vec{v} + q\vec{A},$$

where $\vec{A}$ is the vector potential which is defined by the condition $\mathrm{curl}\vec{A} = \vec{B}$. We can choose $\vec{A} = \frac{1}{2}\hat{\tau}Br$, where r denotes the distance from the z-axis, and $\hat{\tau}$ stands for the unit vector perpendicular both to the radius vector and to the z-axis. This is a vector field obeying cylindrical symmetry around the z-axis and because of that, the corresponding generalized momentum is conserved,

$$\mathcal{L}_z = \vec{R} \times \vec{\mathcal{P}} \cdot \hat{z} = mrv_\perp + \frac{1}{2}qBr^2 \equiv \frac{1}{2}qBR^2;$$

here $v_\perp$ denotes the velocity component perpendicular to the z-axis. Noting that at the topmost position of the ball's trajectory, $v_\perp = v$ due to the fact that the speed is constant and horizontal (i.e. in $x-y$-plane), we obtain a quadratic equation for $\rho = \frac{r}{R}$:

$$\rho^2 + 2\kappa\rho - 1 = 0, \quad \kappa \equiv \frac{mv}{qBR}.$$

We select the positive root $\rho = \sqrt{\kappa^2+1} - \kappa$. Now we can write the final answer as

$$z_{\max} = R\sqrt{1-\rho^2} = R\kappa\sqrt{2}\sqrt{\sqrt{\kappa^{-2}+1}-1}$$

which coincides with the previous result.

Consider three orthogonal unit vectors: 1-st is along with the velocity, 2-nd along with direction from ball to the middle of the rod, and 3-rd is perpendicular to the other two.

As we found earlier that $v = \mathrm{const}$, so $a_1 = 0$. Further, a_2 coinsides with v^2/R because ball is moving around sphere. Finally, the projection of 2-nd Newton law onto the 3-rd axis is: $ma_3 = qvB\frac{z_{\max}}{R}$. Hence

$$a = \sqrt{\left(\frac{v^2}{R}\right)^2 + \left(\frac{qvBz_{\max}}{mR}\right)^2}$$

A1 0.50 At what angle α_1 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be at maximum?

Power of the friction force is given by the equation:

$$P = (F, v) = -\left(\mu_x mg\cos^2\alpha + \mu_y mg\sin^2\alpha\right) v$$

since $\mu_x > \mu_y$, then maximum of the power is reached when $\alpha = 0$

Ответ:

$$\alpha = 0$$

A2 0.50 At what angle α_2 to the X axis should the body velocity vector be for the absolute value of the power of the friction force be 1.2 times less than maximum?

From the previous point it follows that:

$$\mu_x mgv\cos^2\alpha + \mu_y mgv\sin^2\alpha = \frac{1}{1.2}\mu_x mgv$$

From where we get:

$$\cos\alpha_2 = \pm\sqrt{\frac{5\mu_x - 6\mu_y}{6(\mu_x - \mu_y)}} = \pm\frac{1}{\sqrt{2}}$$
$$\sin\alpha_2 = \pm\sqrt{\frac{\mu_x}{6(\mu_x - \mu_y)}} = \pm\frac{1}{\sqrt{2}}$$

Ответ:

$$\alpha_2 = \pm\pi/4; \pm3\pi/4$$

A3 1.00 Let the initial velocity have components $v_{0x} = 1\text{m/s}$ and $v_{0y} = 1\text{m/s}$. After some time the velocity component along the Y axis equals $v_{1y} = 0{,}25\text{m/s}$. What is the velocity magnitude at this moment?

Newton's equation:

$$\begin{cases} m\dfrac{dv_x}{dt} &= -\mu_x mg\dfrac{v_x}{v} \\ m\dfrac{dv_y}{dt} &= -\mu_y mg\dfrac{v_y}{v} \end{cases}$$

Dividing one equation by another, we get:

$$\frac{dv_x}{dv_y} = \frac{\mu_x v_x}{\mu_y v_y}$$

Integrating these equations, we obtain:

$$\frac{v_x^{\mu_y}}{v_y^{\mu_x}} = Const$$

From the initial conditions we find:

$$v_x = 0.125\text{m/s}$$

So the absolute value of velosity is

Ответ:

$$v = 0.28\text{m/s}$$

A4 [1.00] Let the velocity be $v_2 = 1.0 \mathrm{m/s}$. At what angle α_3 to the X axis should the velocity vector be for the radius of curvature of the trajectory be minimum? What is this radius equal to? The free fall acceleration is $g = 9{,}8 \mathrm{m/s}^2$.

Projection of the friction force onto the direction perpendicular to the velocity is given by equation:

$$F_n = mg(\mu_x - \mu_y)\sin\alpha\cos\alpha$$

Newton's second law projected onto the direction perpendicular to the velocity has the form:

$$\frac{mv^2}{R} = mg(\mu_x - \mu_y)\sin\alpha\cos\alpha,$$

where R is the radius of curvature of the trajectory. It's obvious that

$$R = \frac{v^2}{g(\mu_x - \mu_y)\sin\alpha\cos\alpha} = \frac{2v^2}{g(\mu_x - \mu_y)\sin 2\alpha}$$

The minimum radius of curvature will be when:

Ответ:

$$\alpha = \pi/4$$

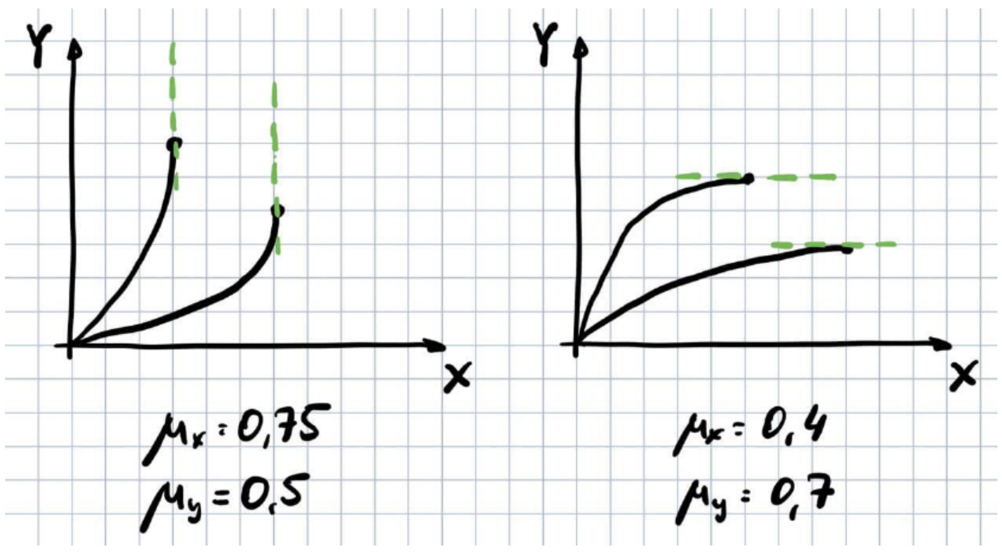
and

$$R_{min} = \frac{2v^2}{g(\mu_x - \mu_y)}$$

A5 [1.00] In a single diagram on the XY plane, sketch the trajectories of the body launched at the angles $\alpha_4 = \pi/6$ and $\alpha_5 = \pi/3$ for the friction coefficients specified above. The magnitudes of initial velocities are the same. Solve the same problem for the friction coefficients $\mu_x = 0{,}4$ and $\mu_y = 0{,}7$.

If $\mu_x = \mu_y$, then the accelerations along the x and y axes will be proportional to the ratio of the initial velocities and the body will move in a straight line. If $\mu_x > \mu_y$, then the speed along the x-axis will decrease faster than in the previous case and the body will deviate from a linear motion as shown in the figure. The direction of deflection does not depend on how the initial velocity is directed. If $\mu_x < \mu_y$, then the body will deviate in the opposite direction.

Ответ:



B1 [2.00] A body of mass m is at rest at the origin. A force has been applied to it at an angle α to the X axis. The force magnitude $F(t) = \gamma t$ linearly grows with time. Find the dependence of the moment the body starts moving on α. Ignore the stagnation phenomenon.

Let us notice that the body will start the motion with the angle $\varphi \neq \alpha$. Then projections of the forces at this moment are given by

$$\begin{cases} F_{\text{fr},x} = \mu_x mg\cos\varphi = \gamma t\cos\alpha \\ F_{\text{fr},y} = \mu_y mg\sin\varphi = \gamma t\sin\alpha \end{cases}$$

Dividing these equations by $\mu_x mg$ and $\mu_y mg$ correspondingly and summing up squares, we get

$$(mg)^2 = (\gamma t)^2\left[\left(\frac{\cos\alpha}{\mu_x}\right)^2 + \left(\frac{\sin\alpha}{\mu_y}\right)^2\right]$$

So the answer is

Ответ:

$$t = \frac{\mu_x \mu_y mg}{\gamma \sqrt{\mu_y^2 \cos^2\alpha + \mu_x^2 \sin^2\alpha}}$$

C1 1.50 For a given initial velocity v_0 find the dependence of its velocity v on the angle of rotation of the rod φ assuming that the other body remains at rest.

Newton's law projected onto the direction of motion of a point-like mass is:

$$m\frac{dv}{dt} = -mg(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)$$

Since $v = L\varphi$, then:

$$L\ddot{\varphi} = -g(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)$$

Multiply this equation by $\dot{\varphi}\,dt$ we get

$$L\dot{\varphi}d\dot{\varphi} = -g(\mu_x \cos^2\varphi + \mu_y \sin^2\varphi)d\varphi$$

Integrating this equation, we obtain:

Ответ:

$$v^2 + gL\left((\mu_x + \mu_y)\varphi + (\mu_x - \mu_y)\frac{\sin 2\varphi}{2}\right) = v_0^2$$

C2 1.50 Find the maximum value of the initial velocity $v_{0\max}$ at which the other body will remain at rest.

Newton's law for a moving body in projection onto a rod has the form:

$$T + F_{\text{fr}} \sin\beta = m\dot{\varphi}^2 L$$

where β is the angle between frictional force and direction of velocity.
Using the result from C1, we find that:

$$T = \frac{mv_0^2}{L} - mg(\mu_x + \mu_y)\varphi - mg(\mu_x - \mu_y)\sin 2\varphi$$

Using the result from B1, we find that:

$$\frac{mv_0^2}{L} - mg(\mu_x + \mu_y)\varphi - mg(\mu_x - \mu_y)\sin 2\varphi \le \frac{\mu_x \mu_y}{\sqrt{\mu_y^2 \sin^2\varphi + \mu_x^2 \cos^2\varphi}}mg$$

Note that with increasing φ, the left hand side decreases, and the right hand side increases, so the body starts to move right at $\varphi = 0$. So

Ответ:

$$v_{0\max} = \sqrt{\mu_y gL} = 2.2\text{m/s}$$

C3 1.00 What distance will the body travel until it stops completely if the initial velocity is $v_{0\max}$?

Substituting $v_{0\max}$ from the previous section in the final equation of the C1, we get the equation for the φ at the stopping point $(v = 0)$:

$$gL\left((\mu_x + \mu_y)\varphi + (\mu_x - \mu_y)\frac{\sin 2\varphi}{2}\right) = \mu_y gL$$

Numerically solving this equation, we get the answer for the φ, and so the travelled distance is equal to

Ответ:

$$L\varphi = 0.34\mathrm{m}$$

A1 1.00 An atom emits light with a wavelength $\lambda_0 = 300\text{nm}$. Using the classical model estimate an emission time τ (that is, the period of time it takes the atom to emit the energy equal to that of a single photon). This time coincides with the characteristic time, during which the atom emits a photon, by the order of magnitude. All radiation is due to a single electron located at a distance about $a_0 = 0.1\text{nm}$ from the nucleus. Express your answer in terms of the physical constants, λ_0, and a_0.

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{p}(t)^2\rangle = \frac{2k}{3c^3}\langle \omega^4 p_m^2 \cos^2(\omega t + \varphi)\rangle = \frac{k\omega^4 p_m^2}{3c^3}.$$

Here p_m is the dipole moment amplitude which can be estimated as $p_m \approx ea_0$. Time of radiation is

$$\tau = \frac{\hbar\omega}{W} \approx \frac{3\hbar}{ka_0^2 e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5 \cdot 10^{-8} s.$$

Here we used the relation $\omega = 2\pi c/\lambda_0$.

Ответ:

$$\tau \approx \frac{3\hbar}{ka_0^2 e^2}\left(\frac{\lambda_0}{2\pi}\right)^3 \approx 1.5 \cdot 10^{-8} s.$$

A2 0.25 Estimate the power W_s of electromagnetic radiation of all N atoms in the spontaneous emission mode, i.e. when the direction of atomic dipole and the phase of its oscillations randomly change from atom to atom. In your answer write down the formula for the power in terms of N, ω, and τ.

System of the atoms' dipole moment is

$$\vec{P}(t) = \sum_i \vec{p}_{mi}\cos(\omega t + \varphi_i).$$

The power of radiation is

$$W = \frac{2k}{3c^3}\langle \ddot{\vec{P}}(t)\rangle = \frac{k\omega^4}{3c^3}\sum_i p_{mi}^2 + \frac{k\omega^4}{3c^3}\sum_{i\neq j}(\vec{p}_{mi}\cdot\vec{p}_{mj})\cos(\varphi_i - \varphi_j).$$

We used $\langle\cos^2(\omega t + \varphi)^2\rangle = \frac{1}{2}$ and $\langle\cos(\omega t + \varphi_1)\cos(\omega t + \varphi_2)\rangle = \frac{1}{2}\cos(\varphi_i - \varphi_j)$. All dipole moments have the same magnitude, are oriented randomly and their phases are not correlated. Therefore the second sum equals zero. The first sum consists of N equal terms and te power is

$$W_s = N\frac{k\omega^4}{3c^3}p_m^2.$$

For one atom we have

$$\frac{k\omega^4 p_m^2}{3c^3} = \frac{\hbar\omega}{\tau}$$

and thus

Ответ:

$$W_s = N\frac{\hbar\omega}{\tau}.$$

A3 0.25 Estimate the duration of the spontaneous emission pulse of this system of atoms. Express your answer in terms of the same quantities.

The energy emitted by all atoms is $E = N\hbar\omega$. Hence the time of radiation is

$$\Delta t_s = \frac{E}{W_s} \approx \tau.$$

Ответ: $\Delta t_s = \tau$.

A4 0.50 Estimate the power W_i of electromagnetic radiation of all N atoms in the superradiance mode, i.e. when the direction of atomic dipoles and the phases of their oscillations are the same for all atoms in the excited state. Express your answer in terms of N, ω, and τ.

In this case all dipole moments are oriented in the same direction and oscillate with the same phase $\varphi_i = \varphi_j$. All terms in both sums in the expression for power in A2 are the same. The first sum has N terms and the second has $N(N-1)$ terms. The power is

$$W_i = (N + N(N-1))\frac{k\omega^4}{3c^3}p_m^2 = N^2\frac{\hbar\omega}{\tau}.$$

Ответ:

$$W_i = N^2\frac{\hbar\omega}{\tau}.$$

A5 0.25 Estimate the duration of the radiation pulse of the system of atoms in the superradiance mode. Express your answer in terms of the same quantities.

Emitted energy is the same as in A3. The time of radiation

$$\Delta t_i = \frac{E}{W_i} \approx \frac{\tau}{N}.$$

Ответ: $\Delta t_i = \frac{\tau}{N}$.

B1 0.50 Let the amplitudes of two wave maxima be E_{m1} and E_{m2}. Find the difference in their propagation speeds Δv. Express your answer in terms of n_0, n_2, c, E_{m1}, and E_{m2}.

Wave propagation speed is

$$v = \frac{c}{n} = \frac{c}{n_0 + n_2E_m^2} \approx \frac{c}{n_0}\left(1 - \frac{n_2E_m^2}{n_0}\right).$$

So the speed difference is

Ответ:

$$\Delta v = v_1 - v_2 = \frac{cn_2}{n_0^2}\left(E_{m2}^2 - E_{m1}^2\right).$$

B2 2.00 A light pulse with a wavelength in vacuum of $\lambda_0 = 300\mathrm{nm}$ and a maximum intensity of $I_0 = 3\cdot 10^9\mathrm{W/cm^2}$ propagates along the axis of a quartz fiber. Assume the envelope of a time dependence of the electric field squared $E_m^2(t)$ of the wave to be a parabola. How far (find the distance s) does the pulse propagate along the fiber before its spectral width increases by the factor of $K = 200$? Express your answer in terms of K, λ_0, n_2, E_m and calculate the numerical value (in meters, rounded to an integer).

First approach.

The intensity is proportional to the square of electric field amplitude. Pulse envelope equation is

$$E_m^2(t) = E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right).$$

Here t_0 is a time when the center of the wavepocket crosses given point in space. Value of t_0 depends on the coordinate along the wave. Electric field in the wave is equal to $E_x(t,\, z) = E_m(t)\cos\varphi(t,\, z)$, where the coordinate z is measured along the direction of wave propagation, and the phase is

$$\varphi(t,\, z) = \omega_0 t - n_0\frac{\omega_0}{c}z - n_2E_m^2(t)\frac{\omega_0}{c}z.$$

We neglect initial frequency difference because spectral width increases by hundreds of times. In this way, nonlinear effects lead to additional phase shift, proportional to the square of $(t - t_0)$:

$$\delta\varphi = n_2E_0^2\left(1 - \frac{4(t-t_0)^2}{(\Delta t_0)^2}\right)\frac{\omega_0}{c}z.$$

Angular frequency is the the rate of phase change with time. So quadratic phase shift leads to linear frequency shift

$$\partial\delta\varphi \quad \omega_0 \quad t - t_0$$

$$\delta\omega(t,\, z) = \frac{\partial\varphi}{\partial t} = -8n_2E_0^2\frac{\omega_0}{c}z\frac{t-t_0}{(\Delta t_0)^2}.$$

Spectral width of the pulse after propagating the distance z in the medium is equal to the difference of frequencies at $t = t_0 \pm \Delta t_0/2$:

$$\Delta\omega = -\frac{8}{\Delta t_0}n_2E_0^2\frac{\omega_0}{c}z\Delta\omega_0.$$

To obtain this formula we used $\Delta\omega_0 \approx 2\pi/\Delta t_0$. The ratio of spectral width at distance $z = s$ and initial spectal width is

$$K = \frac{|\Delta\omega|}{\Delta\omega_0} = \frac{4}{\pi}n_2E_0^2\frac{\omega_0}{c}s,$$

Therefore the required distance is

$$s = \frac{\pi K c}{4n_2E_m^2\omega_0} = \frac{K\lambda_0}{8n_2E_m^2}.$$

To calculate the numerical value of s, one should remember that the peak pulse intensity value is three times higher than the one mentioned in the problem description. Thus the value of $n_2E_m^2$ is three times higher: $n_2E_m^2 \approx 9.6 \cdot 10^{-7}$.

Second approach.

Let us consider the propagation of two adjacent electric field maxima. Due to nonlinearity this maxima move with slightly different speed:

$$\delta v \approx \frac{cn_2}{n_0^2}(E_{m2}^2 - E_{m1}^2).$$

For the given pulse duration and mean frequency it consists of $N = \Delta t_0/T_0 = \omega_0/\Delta\omega_0$ oscillations. The electric field maxima can be enumerated from $-N/2$ to $N/2$. Then the amplitude of electric field maximum with number l is (the maximum with large number arrives at a given point later):

$$E_m^2(l) = E_0^2\left(1 - \frac{4l^2}{N^2}\right).$$

Consequently, the speed difference between adjacent maxima with numbers l and $l+1$ equals

$$\Delta v_l = v_{l+1} - v_l \approx \frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}.$$

Initial frequency is equal ω_0. After propagation over distance s the frequency of the maximum with number l is:

$$\omega_l = \frac{2\pi}{T_l} = \frac{2\pi}{2\pi/\omega_0 + s/v_{l+1} - s/v_l} = \frac{\omega_0}{1 - s\Delta v_l\omega_0/(2\pi v_l^2)} = \omega_0 + \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}E_0^2\frac{8l}{N^2}s.$$

Frequency linearly depends on the maximum number. The maximal frequency difference is between the first and the last maxima and equals

$$\Delta\omega = \frac{\omega_0^2}{2\pi v_l^2}\frac{cn_2}{n_0^2}\frac{8}{N}s.$$

Hence ($v_l \approx v = c/n_0$ and $\omega_0 = N\Delta\omega_0$)

$$K = \frac{\Delta\omega}{\Delta\omega_0} = \frac{8n_2E_m^2}{\lambda_0}s,$$

and the answer is the same.

Ответ:

$$s = \frac{K\lambda_0}{8n_2E_m^2} \approx 7.8m.$$

B3 0.50 What sign should the constant β_2 have in order for the pulse chirped according to the scheme described above to be compressed in time in this medium? Please, indicate "+" or "-" in your answer. In what follows consider that β_2 has exactly this sign.

To squeeze the chirping pulse, the relatively high-frequency tail of the pulse should propagate faster than its low-frequency 'head'. This is called 'anomalous dispersion'. Speed of the wave propagation is the group velocity

$$V_g = \frac{d\omega}{dk} = \frac{1}{\beta_1 + \beta_2(\omega - \omega_0)},$$

Anomalous dispersion (larger group speed at larger frequencies) is observed for $\beta_2 < 0$.

Ответ: $\beta_2 < 0$.

Условие

B4 1.00 A pulse described in B2 has a duration $\Delta t_0 = 10\text{ps}$ and an initial spectral width $\Delta\omega_0 \approx 2\pi/\Delta t_0$ (before chirping) and propagates in the medium described above. Find the distance the pulse should travel in order to achieve the minimum possible duration after chirping with spectrum broadening by the factor of $K = 200$. Express your answer in terms of physical constants, $K, \Delta t_0, \beta_1$, and β_2 and calculate the numerical value in meters, rounded to an integer.

Since the term associated with β_2 is much less in absolute value than β_1 for the given range of frequencies, we can use approximate expression for the group velocity

$$V_g \approx \frac{1}{\beta_1} - \frac{\beta_2}{\beta_1^2}(\omega - \omega_0).$$

The length of the chirping pulse that just has entered into the considered medium is $\Delta l = V_g \Delta t_0 = \frac{1}{\beta_1}\Delta t_0$. The difference in group velocity of the 'tail' and the 'head' is

$$\Delta V_g \approx \frac{|\beta_2|}{\beta_1^2}\Delta\omega = \frac{|\beta_2|}{\beta_1^2}K\frac{2\pi}{\Delta t_0}.$$

The time in which the 'tail' will catch up with the 'head' up (which corresponds to the minimal duration of the pulse) is

$$t = \frac{\Delta l}{\Delta V_g} = \frac{\beta_1}{2\pi K|\beta_2|}(\Delta t_0)^2.$$

Since $V_g \approx \frac{1}{\beta_1}$ in this period of time the pulse will pass the distance

$$l = V_g t \approx \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

Ответ:

$$l = \frac{1}{2\pi K|\beta_2|}(\Delta t_0)^2 \approx 4m.$$

B5 1.50 Nonlinearity of a medium leads to disappearance of diffraction of a light beam of sufficiently high intensity. Estimate the minimum power of a light pulse W_c at which it does not experience diffraction, i.e. propagates inside a narrow cylindrical channel of constant radius. Express your answer for W_c in terms of physical constants, frequency ω_0, n_0, and n_2. Assume the intensity distribution over the channel cross section to be approximately uniform. Find the numerical value of the power for a pulse with a wavelength in vacuum $\lambda_0 = 300\text{nm}$ propagating in quartz. Coefficient $n_0 = 1.47$.

Let be a radius of the channel. Diffraction divergence angle of the beam can be estimated as

$$\theta_d \approx \frac{\lambda}{\pi a} = \frac{2c}{n_0\omega_0 a}.$$

However, due to nonlinearity, the refractive index inside the channel $n = n_0 + n_2E_m^2$ is larger than refractive index $n = n_0$ outside of it. The beam deflecting from the channel's axis due to diffraction will not get out of it if its angle of incidence will be greater than the angle of total internal reflection α_c. So the critical power corresponds to the condition $\theta_d = \frac{\pi}{2} - \alpha_c \equiv \theta_c$.
Since

$$\cos\theta_c \approx 1 - \frac{\theta_c^2}{2} = \sin\alpha_c = \frac{n_0}{n_0 + n_2E_m^2} \approx 1 - \frac{n_2E_m^2}{n_0},$$

we have the following condition

$$E_m^2 = \frac{n_0}{2n_2}\theta_d^2 = \frac{2c^2}{n_0n_2\omega_0^2}\frac{1}{a^2}.$$

Corresponding value of the intensity is

$$I_c = \frac{\varepsilon_0 n_0 c}{2}E_m^2 = \frac{\varepsilon_0 c^3}{n_2\omega_0^2 a^2}.$$

Finally, 'critical' power is

$$W_c = I\pi a^2 = \frac{\pi\varepsilon_0 c^3}{n_2\omega_0^2}$$

and does not depend on the channel diameter.
The numerical value of the power for the pulse of wavelength λ_0 in a vacuum is equal to

$$W_c = \frac{2\pi n_0\lambda_0^2 I_1}{3.2\cdot 10^{-7}} \approx 26MW.$$

Comment: The expression for the diffraction divergence of the circular beam is approximate. Therefore, it is possible to use alternative estimations, for instance $\theta_d \approx \lambda/a$, or $\theta_d \approx \lambda/2\pi a$.

Ответ:

$$W_c = \frac{\pi \varepsilon_0 c^3}{n_2 \omega_0^2} = \frac{2\pi n_0 \lambda_0^2 I_1}{3.2 \cdot 10^{-7}} \approx 26MW.$$

C1 1.00 Propose a method that would allow one to detect an exoplanet with a noticeable inclination of its orbital plane with respect to the line of sight by means of studying the spectrum of its star in the optical range. As an answer name the physical phenomenon underlying your method.

The most natural way is to detect the Doppler shift of some element (e.g. hydrogen) spectral line. The shift is caused by the star rotation with respect to the star-exoplanet system barycenter. This shift should be observed with the exoplanet rotation period.

Ответ: Doppler effect

C2 1.00 Suppose an exoplanet of mass m orbits a star of mass M in a circular orbit of a radius R and the period of revolution is T. The orbital plane is at an angle θ to the direction to Earth. Estimate the accuracy of the relative frequency measurement, $\Delta\omega/\omega$, required to detect such an exoplanet by your method. In your answer express $\Delta\omega/\omega$ in terms of the fundamental constants, R, T, θ, m, and M.

Planet's orbital motion velocity is $v = \frac{2\pi R}{T}$. The corresponding velocity of the star is $v_1 = \frac{vm}{M}$. The Doppler effect is sensitive to the velocity sightline component only (the nonrelativistic case is considered). Its maximum value is $v_1 \cos(\theta)$. Thus the relative frequency shift is

Ответ:

$$\frac{\Delta\omega}{\omega} = \frac{2\pi R}{Tc}\frac{m}{M}\cos\theta.$$

C3 0.25 Assume the mass of the exoplanet and its star to be equal to the mass of Earth and the Sun, respectively. Assume the radius of the circular orbit to be equal to the distance from Earth to the Sun ($R \approx 1.5 \cdot 10^{11}\mathrm{m}$), the angle $\theta = 60^\circ$. The Solar mass is 330,000 times of the Earth's mass, the period of the Earth's revolution around the Sun is 1 year. Find an integer n such that 10^{-n} is the accuracy of relative frequency measurement required by your method. Usage of ultrashort (femtosecond) laser pulses makes it possible to measure frequencies in the optical range ($10^{15}\mathrm{Hz}$) with an accuracy of about 10 Hz. Is this accuracy enough to register the exoplanet?

Even for $\theta = 60^o$ the star velocity sightline component is at most half v_1. In order to estimate this component, one could assume the exoplanet orbit is circular. Since the same assumption is valid for the Earth's orbit, the maximal sightline projection is

$$v_m = \frac{2\pi R}{T}\frac{1}{330000}\cos\theta \approx 4.5\frac{sm}{s}.$$

The corresponding frequency shift is of order $\frac{\Delta\omega}{\omega} \approx \frac{v_m}{c} \approx 10^{-10}$. Thus, a frequency variation in the tenth significant figure should be detected once a year. The accuracy of an optical frequency comb is of order $\frac{10Hz}{10^{15}Hz} \approx 10^{-14}$. So, exoplanet detection is quite possible.

This method is indeed applied in practice. However a number of difficulties had to be overcome to actually use this technology. In fact, most of the stably generated femtosecond pulses have carrier frequencies outside the visible spectrum, in the microwave and infrared regions. In order to use the method in visible spectrum, a different nonlinear phenomenon should be used -- higher harmonic generation in nonlinear medium. Moreover, to increase the spectral device resolution, frequency filtering should be used which increases the narrow frequency intervals between the lines.

Ответ: $n = 10$, exoplanet detection is possible.