Characterization of Soil Colloids (10 points)

Colloidal science is useful to characterize soil particles. For example, Brownian motion (random motion of colloidal particles) can be used to measure particle sizes.

Part A. Motions of colloidal particles (1.6 points)

We analyze the one-dimensional Brownian motion of a colloidal particle with mass M. The equation of motion for its velocity $v(t)$ reads:

$$M\dot{v} = -\gamma v(t) + F(t) + F_{\mathrm{ext}}(t), \tag{1}$$

where γ is the friction coefficient, $F(t)$ is a force due to random collisions of water molecules, and $F_{\mathrm{ext}}(t)$ is an external force. In Part A, we assume $F_{\mathrm{ext}}(t) = 0$.

A.1	Consider that a water molecule collides with the particle at $t = t_0$, giving impulse I_0, and $F(t) = 0$ afterward. If $v(t) = 0$ before the collision, $v(t) = v_0 e^{-(t-t_0)/\tau}$ for $t > t_0$. Determine v_0 and τ.	0.8pt

In the following, you may use τ in your answers.

A.2	Actually, water molecules collide with the particle one after another. Determine $v(t)$ on condition that $v(0) = 0$ and the ith collision gives the impulse I_i at time t_i.	0.8pt

Part B. Effective equation of motion (1.8 points)

Results so far indicate that particle velocities $v(t)$ and $v(t')$ are uncorrelated and random if $|t - t'| \gg \tau$. Therefore, one-dimensional Brownian motion can be approximated by random changes in the velocity at each time interval δ $(\gg \tau)$, i.e.,

$$v(t) = v_n \quad (t_{n-1} < t \leq t_n), \tag{2}$$

with $t_n = n\delta$ $(n = 0, 1, 2, \cdots)$ and a random number v_n. It satisfies

$$\langle v_n \rangle = 0, \quad \langle v_n v_m \rangle = \begin{cases} C & (n = m), \\ 0 & (n \neq m), \end{cases} \tag{3}$$

with a parameter C depending on δ. Here $\langle X \rangle$ indicates the expectation value of X. That is, if you draw random numbers X infinite times, the mean will be $\langle X \rangle$.

Now we consider the particle displacement $\Delta x(t) = x(t) - x(0)$ for $t = N\delta$ with an integer N.

B.1	Determine $\langle \Delta x(t) \rangle$ and $\langle \Delta x(t)^2 \rangle$ using C, δ, and t.	1.0pt

B.2	The quantity $\langle \Delta x(t)^2 \rangle$ is called the mean square displacement (MSD). It is a characteristic observable of the Brownian motion, which corresponds to the limiting case $\delta \to 0$. From this, we can show $C \propto \delta^\alpha$ and $\langle \Delta x(t)^2 \rangle \propto t^\beta$. Determine α and β.	0.8pt

Part C. Electrophoresis (2.7 points)

Here we discuss electrophoresis, i.e., transport of charged particles by an electric field. Suspension of colloidal particles with mass M and charge $Q\ (>0)$ is put in a narrow channel with a cross-section A (Fig.1(a)). We ignore the interaction between particles, effects of counter-ions, and gravity.

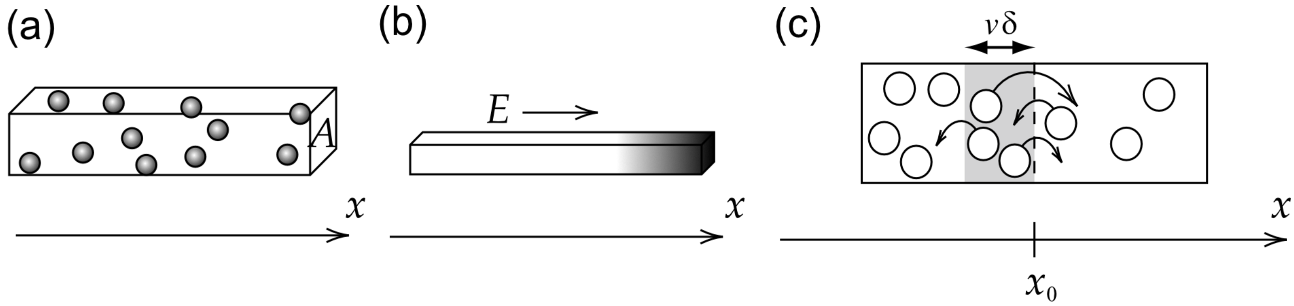


Fig.1: Setting for Part C.

By applying an electric field E in the x-direction, particles are transported and their concentration $n(x)$ (particle number per unit volume) becomes non-uniform (Fig.1(b)). When E is removed, this non-uniformity gradually disappears. This is due to Brownian motion of particles. If $n(x)$ is not uniform, the numbers of right-going and left-going particles may differ (Fig.1(c)). This generates a particle flux $J_D(x)$, the number of particles flowing at x along the x-axis per unit cross-sectional area and unit time. This flux is known to satisfy

$$J_D(x) = -D\frac{dn}{dx}(x), \tag{4}$$

where D is called the diffusion coefficient.

Now let's consider the particles having velocities $\pm v$ with a given $v(>0)$. The proportions of particles with velocity $+v$ and $-v$ are the same and denoted by $p(v)$, hence the concentration is $p(v)n(x)$. Let $j_\pm(x,v)$ be the flux of particles with velocities $\pm v$ and consider $j_D(x_0,v) = j_+(x_0,v) - j_-(x_0,v)$. For particles with velocity $+v$ to cross x_0 in the time interval δ, they should be in the shaded region of Fig.1(c). Since δ is small, we have $n(x) \simeq n(x_0) + (x-x_0)\frac{dn}{dx}(x_0)$ in this region.

C.1	Determine $j_D(x_0,v)$ using necessary quantities from v, δ, $p(v)$, $n(x_0)$, and $\frac{dn}{dx}(x_0)$.	0.7pt

C.2	The total flux $J_D(x)$ is given by $J_D(x) = \langle \frac{j_D(x,v)}{2p(v)} \rangle$. Using this and Eq.(4), express D in terms of C and δ, and $\langle \Delta x(t)^2 \rangle$ in terms of D and t.	0.5pt

Now we discuss the effect of osmotic pressure Π. It is given by $\Pi = \frac{n}{N_A}RT = nkT$ with the Avogadro constant N_A, the gas constant R, temperature T, and the Boltzmann constant $k = \frac{R}{N_A}$. Let us consider the non-uniform concentration formed under the electric field E (Fig.1(b)). Since $n(x)$ depends on x, so does $\Pi(x)$. Then the forces due to $\Pi(x)$ and $\Pi(x+\Delta x)$ must be balanced with the total force from the field E acting on the particles (Fig.2). Here we consider small Δx, so that $n(x)$ can be regarded as constant over this range, while $n(x+\Delta x) - n(x) \simeq \Delta x \frac{dn}{dx}(x)$.

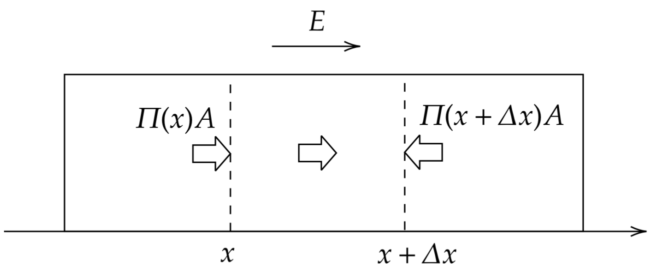


Fig.2: Force balance.

C.3	Express $\frac{dn}{dx}(x)$ using $n(x)$, T, Q, E, and k.	0.5pt

Let us discuss the balance of the flux now. Besides the flux $J_D(x)$ due to the Brownian motion, there is also a flux due to the electric field, $J_Q(x)$. It is given by

$$J_Q(x) = n(x)u, \tag{5}$$

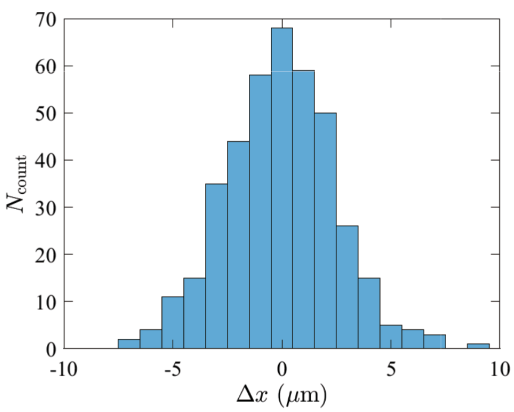
where u is the terminal velocity of particles driven by the field.

C.4	To determine u, we use Eq.(1) with $F_{\text{ext}}(t) = QE$. Since $v(t)$ is fluctuating, we consider $\langle v(t)\rangle$. Assuming $\langle v(0)\rangle = 0$ and using $\langle F(t)\rangle = 0$, evaluate $\langle v(t)\rangle$ and obtain $u = \lim_{t\to\infty}\langle v(t)\rangle$.	0.5pt

C.5	The flux balance reads $J_D(x) + J_Q(x) = 0$. Express the diffusion coefficient D in terms of k, γ, and T.	0.5pt

Part D. Mean square displacement (2.4 points)

Suppose we observed the Brownian motion of an isolated, spherical colloidal particle with radius $a = 5.0\ \mu$m in water. Figure 3 shows the histogram of displacements Δx measured in the x-direction at every interval $\Delta t = 60$ sec. The friction coefficient is given by $\gamma = 6\pi a\eta$ with water viscosity $\eta = 8.9 \times 10^{-4}$ Pa · s and the temperature was $T = 25\ ^\circ$C.



Δx (μm)	−10	−9	−8	−7	−6	−5	−4
N_{count}	0	0	0	2	4	11	15

Δx (μm)	−3	−2	−1	0	1	2	3
N_{count}	35	44	58	68	59	50	26

Δx (μm)	4	5	6	7	8	9	10
N_{count}	15	5	4	3	0	1	0

Fig.3: Histogram of displacements.

D.1	Estimate the value of N_A without using the fact that it is the Avogadro constant, up to two significant digits from the data in Fig.3. The gas constant is $R = 8.31$ J/K $\cdot$ mol.	1.0pt

Now we extend the model in Part B to describe the motion of a particle with charge Q under an electric field E. The particle velocity $v(t)$ considered in Eq.(2) should be replaced by $v(t) = u + v_n \ (t_{n-1} < t \le t_n)$ with v_n satisfying Eq. (3) and u being the terminal velocity considered in Eq.(5).

D.2	Express the MSD $\langle \Delta x(t)^2 \rangle$ in terms of u, D, and t. Obtain approximate power laws for small t and large t, as well as the characteristic time t_* where this change occurs. Draw a rough graph of MSD in a log-log plot, indicating the approximate location of t_*.	0.8pt

Next, we consider swimming microbes (Fig.4(a)), in one dimension for simplicity (Fig.4(b)). These are spherical particles with radius a. They swim at velocity either $+u_0$ or$-u_0$, the sign chosen randomly at every time interval δ without correlation. The observed motion is a combination of displacements due to swimming and those due to the Brownian motion of a spherical particle.

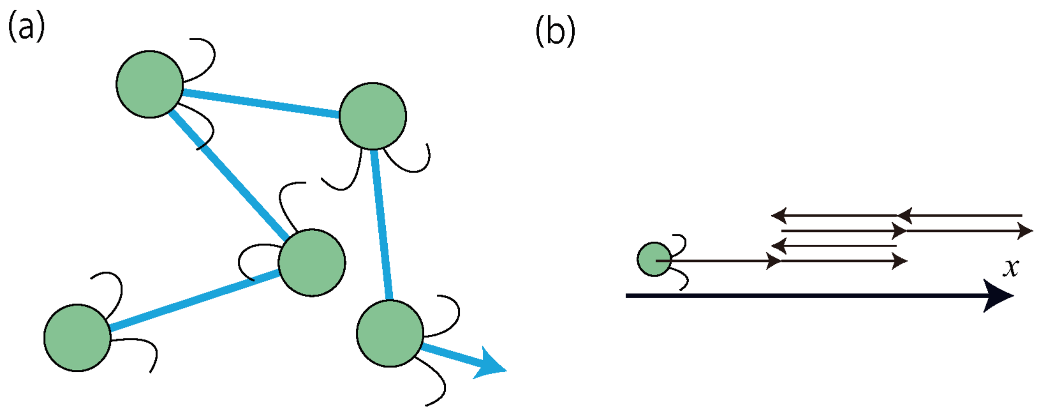


Fig.4: (a) Motion of microbes. (b) Its one-dimensional version.

D.3 Figure 5 shows the MSD $\langle \Delta x(t)^2 \rangle$ of those microbes, showing $\langle \Delta x(t)^2 \rangle \propto t$ for small and large t, and $\langle \Delta x(t)^2 \rangle \propto t^2$ in between. Obtain the power law with the prefactor for each time range. **0.6pt**

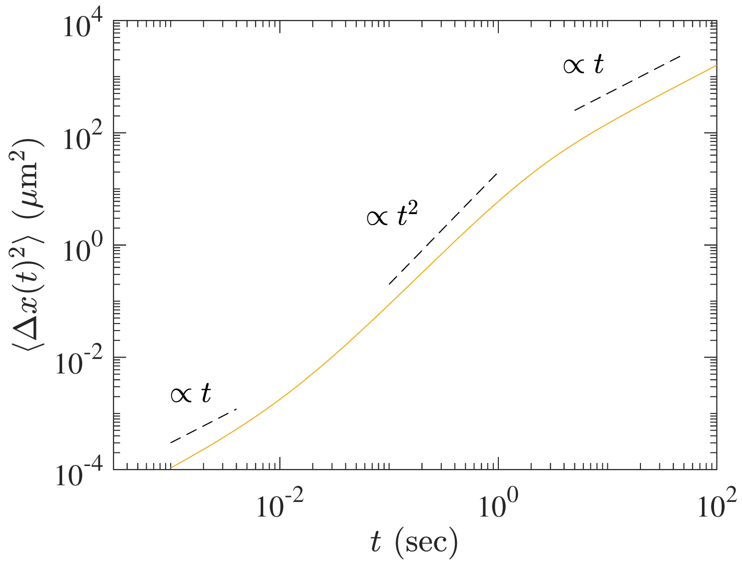


Fig.5: Mean square displacement of the microbes.

Part E. Water purification (1.5 points)

Here we discuss the purification of water including soil particles, by adding electrolytes to coagulate them. Particles interact through van der Waals force and electrostatic force, the latter including effects of both surface charges and the surrounding layer of counter-ions (called the double layer; Fig.6(a)). As a result, the interaction potential for particle distance d (Fig.6(b)) is given by

$$U(d) = -\frac{A}{d} + \frac{B\epsilon(kT)^2}{q^2} e^{-d/\lambda}, \tag{6}$$

where A and B are positive constants, ϵ is the dielectric constant of water, and λ is the thickness of the double layer. Assuming that charges of ions are $\pm q$, we have

$$\lambda = \sqrt{\frac{\epsilon kT}{2N_A q^2 c}} \tag{7}$$

with the molar concentration of ion, c.

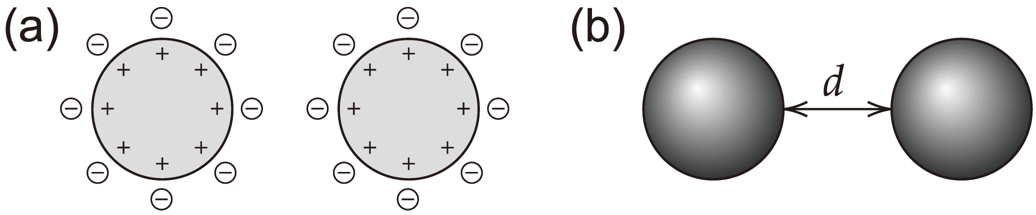


Fig.6: (a) Surface charges of colloidal particles and counter-ions. (b) Definition of the distance d.

E.1	Addition of sodium chloride (NaCl) to the suspension causes colloidal particles to coagulate. Determine the lowest concentration c of NaCl necessary for coagulation. It is sufficient to consider two particles without thermal fluctuations, i.e., $F(t) = 0$ in Eq.(1), and assume that the terminal velocity for the given potential force is reached instantaneously.	1.5pt

Neutron Stars (10 points)

We discuss the stability of large nuclei and estimate the mass of neutron stars theoretically and experimentally.

Part A. Mass and stability of nuclei (2.5 points)

The mass of a nucleus $m(Z,N)$ consisting of Z protons and N neutrons is smaller than the sum of masses of protons and neutrons, hereafter called nucleons, by the binding energy $B(Z,N)$. Ignoring minor corrections, we can approximate the binding energy consisting of the volume term with a_V, the surface term with a_S, the Coulomb energy term with a_C, and the symmetry energy term with a_{sym} in the following way.

$$m(Z,N)c^2 = Am_N c^2 - B(Z,N), \qquad B(Z,N) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_{\text{sym}} \frac{(N-Z)^2}{A}, \tag{1}$$

where $A = Z + N$ is the mass number, c is the speed of light in vacuum, and m_N is the nucleon mass. In the calculation, use $a_V \approx 15.8\ \mathrm{MeV}$, $a_S \approx 17.8\ \mathrm{MeV}$, $a_C \approx 0.711\ \mathrm{MeV}$, and $a_{\text{sym}} \approx 23.7\ \mathrm{MeV}$ (MeV = 10^6 electron volts).

A.1	Under the approximation of $Z = N$, determine A for maximizing the binding energy per nucleon, B/A.	0.9pt

A.2	Under the condition of fixed A, the atomic number of the most stable nucleus Z^* is determined by maximizing $B(Z, A-Z)$. For $A = 197$, calculate Z^* using Eq. (1).	0.9pt

A.3 (0.7pt)

A nucleus having large A breaks up into lighter nuclei through fission in order to minimize the total rest-mass energy. When the following energy relation holds,

$$m(Z,N)c^2 > 2m(Z/2, N/2)c^2,$$

a nucleus with (Z,N) with both Z and N being even numbers can break up into two nuclei with $(Z/2, N/2)$. When this relation is written as

$$Z^2/A > C_{\text{fission}} \frac{a_S}{a_C},$$

obtain C_{fission} up to two significant digits.

Part B. Neutron star as a gigantic nucleus (1.5 points)

For large nuclei with a large enough mass number $A > A_c$ with a threshold A_c, these nuclei stay stable against nuclear fission because of the sufficiently large binding energy due to gravity.

B.1	We assume that $N = A$ and $Z = 0$ is realized for sufficiently large A and Eq. (1) is not modified. The binding energy due to gravity is $$B_{\text{grav}} = \frac{3}{5}\frac{GM^2}{R},$$ where $M = m_N A$ and $R = \gamma A^{1/3}$ with $\gamma \simeq 1.1 \times 10^{-15}\ \mathrm{m} = 1.1\ \mathrm{fm}$ are the mass and the size of the nucleus, respectively. For $B_{\text{grav}} = a_{\text{grav}} A^{5/3}$, obtain a_{grav} in the MeV unit up to the first significant digit. Then, ignoring the surface term, estimate A_c up to the first significant digit. In the calculation, use $m_N c^2 \simeq 939\ \mathrm{MeV}$ and $G = \hbar c / M_P^2$ where $M_P c^2 \simeq 1.22 \times 10^{22}\ \mathrm{MeV}$ and $\hbar c \simeq 197\ \mathrm{MeV} \cdot \mathrm{fm}$.	1.5pt

Part C. Neutron star in a binary system (6.0 points)

Some neutron stars are pulsars regularly emitting electromagnetic waves, which we call "light" for simplicity here, at a constant period. Neutron stars often make binary systems with a White Dwarf. Let us consider the star configuration shown in Fig. 1, where a light pulse from a neutron star **N** to the Earth **E** passes near a White Dwarf **W** of the binary system. Measuring these pulses influenced by the star's gravity leads to an accurate estimation of the mass of **W** as explicated below, resulting in the estimation of the mass of **N**.

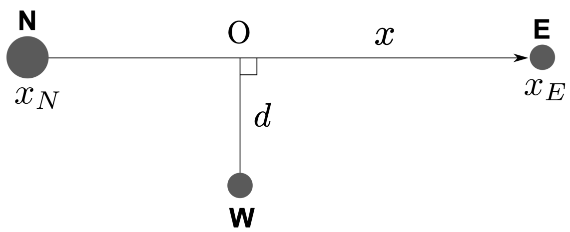
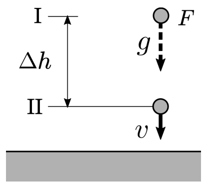


Fig. 1: Configuration of light pulse path.

C.1 As shown in the figure below, under the constant gravitational acceleration g we place two levels I and II with the height difference Δh. Set the identical clocks at I, II, and F, the free-falling system, denoted by clock-I, clock-II, and clock-F, respectively. 1.0pt



Set-up of the thought experiment.

Suppose that time $\Delta\tau_{\mathrm{I}}$ ticks by on the clock-I while time $\Delta\tau_{\mathrm{II}}$ elapses on the clock-II. We assume that initially F is placed at the same height as that of I and its velocity is zero, and we adjust the clock-F and the clock-I so that their unit time length is $\Delta\tau_{\mathrm{I}}$. Then we let the F fall freely and pass II with a velocity v. Under the assumption that time on the clocks of the free-falling system elapses without the influence of gravity, one time unit of the clock-F remains to be $\Delta\tau_{\mathrm{I}}$. Seen from F, II is relatively moving upward with the velocity v at this moment, so that the time dilation can be estimated by the Lorentz transformation.
Determine $\Delta\tau_{\mathrm{II}}$ in terms of $\Delta\tau_{\mathrm{I}}$ up to the first order in $\Delta\phi/c^2$, where $\Delta\phi = g\Delta h$ is a difference of the gravitational potential, *i.e.*, the gravitational potential energy per unit mass.

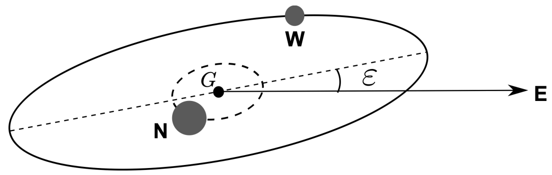
C.2 Under the gravitational potential ϕ time delays change the effective speed of light. When $\phi(r = \infty) = 0$, the effective speed of light, c_{eff}, observed at the infinity can be given up to the first order in ϕ/c^2 as 1.8pt

$$c_{\text{eff}} \approx \left(1 + \frac{2\phi}{c^2}\right) c$$

including the effect of space distortion. We note that the light path can be approximated as a straight line.
As shown in Fig. 1, we take the x-axis along the light path from the neutron star **N** to the Earth **E** and place $x = 0$ at the point where the White Dwarf **W** is the closest to the light path. Let $x_N\ (< 0)$ be the x-coordinate of **N**, $x_E\ (> 0)$ be that of **E**, and d be the distance between **W** and the light path.
Estimate the changes of the arrival time Δt caused by the White Dwarf with mass M and express the answer in a simple form disregarding higher order terms of the following small quantities: $d/|x_N| \ll 1$, $d/x_E \ll 1$, and $GM/(c^2 d) \ll 1$. If necessary, use the following formula.

$$\int \frac{dx}{\sqrt{x^2+d^2}} = \frac{1}{2}\log\left(\frac{\sqrt{x^2+d^2}+x}{\sqrt{x^2+d^2}-x}\right) + C.$$

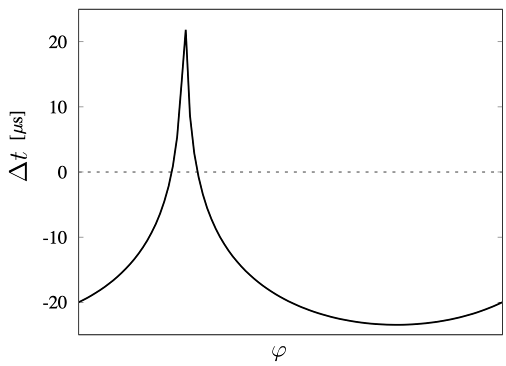
C.3 As shown below, in a binary star system **N** and **W** are assumed to be moving in circular orbits with zero eccentricity around the center of mass G on the orbit plane. Let ε be the orbital inclination angle measured from the orbit plane to the line directed toward **E** from G, and let L be the length between **N** and **W** and M_{WD} be the mass of the White Dwarf. In the following, we assume $\varepsilon \ll 1$. 1.8pt



Binary star system.

We observe light pulses from **N** on **E** far away from **N**. The light path to **E** varies with time depending on the configuration of **N** and **W.** The delay in the time interval of arriving pulses on **E** has the maximum value Δt_{max} for $x_N \simeq -L$ and the minimum value Δt_{min} for $x_N \simeq L$. Calculate $\Delta t_{\text{max}} - \Delta t_{\text{min}}$ in a simple form disregarding higher order terms of small quantities as done in **C.2**. We note that the delays due to gravity from stellar objects other than **W** are assumed to cancel out in $\Delta t_{\text{max}} - \Delta t_{\text{min}}$.

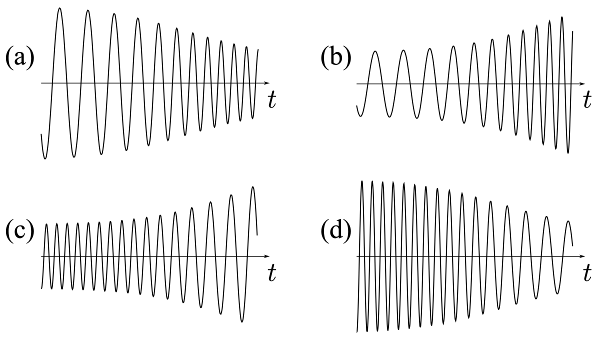
C.4 The below figure shows the observed time delays as a function of the orbital phase φ for the binary star system with $L \approx 6 \times 10^6\,\mathrm{km}$ and $\cos\varepsilon \approx 0.99989$. Estimate M_{WD} in terms of the solar mass $M_\odot$ and show the results for $M_{\text{WD}}/M_\odot$ up to the first significant digit. Here the approximate relation, $GM_\odot/c^3 \approx 5\,\mu\mathrm{s}$, can be used. 0.8pt



Observed time delays Δt as a function of the orbital phase φ to locate **N** and **W** on the orbits.

C.5 In the binary system of neutron stars, two stars release energy and angular momentum by emitting gravitational waves and eventually collide to merge. For simplicity, let us consider only a circular motion with the radius R and the angular velocity ω and then $\omega = \chi R^p$ holds with the constant χ depending on neither ω nor R. Determine the value for p. 0.4pt

C.6 The amplitude of the emitted gravitational wave from the binary system in **C.5** is proportional to $R^2\omega^2$. Figure below qualitatively shows four different temporal profiles of the observed gravitational waves before the two-star collision. Select the most appropriate profile from (a) to (d). 0.2pt



Observed data profiles of gravitational waves.

Water and Objects (10 pt)

In this problem, we consider the phenomena caused by the interaction between water and objects.

If necessary, you can use the fact that if the function $y(x)$ satisfies the differential equation $y''(x) = ay(x)$ (a is a positive constant), then its general solution is $y(x) = A\exp(\sqrt{a}x) + B\exp(-\sqrt{a}x)$, where A and B are arbitrary constants.

Part A. Merger of water drops (2.0 points)

We consider two stationary, spherical water drops on the surface of a strong hydrophobic material, shown in Fig.1.

Initially neighboring two spherical water drops with the same radius are placed on the surface; then these two drops are merged after touching each other and form a larger spherical water drop, which suddenly jumps up.

A.1	The radius a of both water drops before the merger is $100\ \mu$m. The density of water ρ is 1.00×10^3 kg/m^3. The surface tension γ is 7.27×10^{-2} J/m^2. Then, determine the initial jump-up velocity, v, of the merged water drop in two significant digits under the following assumptions: • Six percent of ΔE, the surface energy difference, is transformed into the kinetic energy of the jumped water drop. • Before and after merger, the total volume of water is conserved.	2.0pt

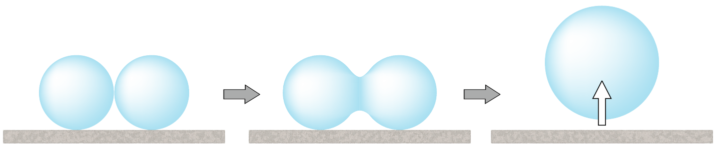


Fig. 1: Merger of two water drops and jump of the merged water drop.

Part B. A vertically placed board (4.5 points)

A flat board is immersed vertically in water. Figures 2(a) and 2(b) respectively show water surface forms for the hydrophilic and hydrophobic board materials. We neglect the thickness of the board.

The board surface is on the yz plane, and the horizontal water surface far away from the board is on the xy plane. The surface shape does not depend on the y-coordinate. Let $\theta(x)$ be the angle between the water surface and the horizontal plane at a point (x, z) on the water surface in the xz plane. Here $\theta(x)$ is measured with respect to the positive x axis and the counterclockwise rotation is taken as positive. Let $\theta(x)$ be θ_0 at the point of contact between the board and the water surface ($x = 0$).

Water density ρ is constant and water surface tension γ is uniform. The gravitational acceleration constant is given by g. Let us determine the water surface form in the following steps.

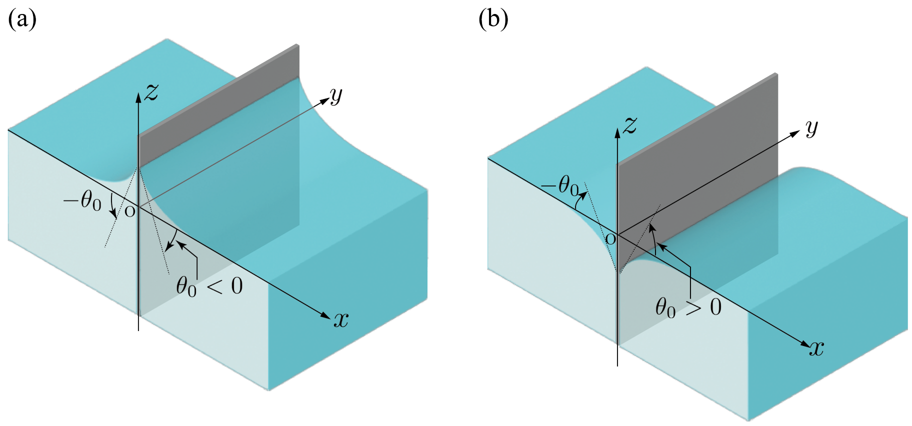


Fig. 2: Boards vertically immersed in the water. (a) hydrophilic board case; (b) hydrophobic board case.

B.1 We consider a hydrophilic board case, as shown in Fig.2(a). The atmospheric pressure, P_0, is assumed to be always uniform. We note that the water pressure, P, satisfies the conditions $P < P_0$ for $z > 0$ and $P = P_0$ for $z = 0$. Then, express P at z in terms of ρ, g, z, and P_0. **0.6pt**

B.2 We consider a water block whose cutout is shown as shaded in Fig.3(a). Its xz plane cross-section is shown in a hatched area in Fig.3(b). Let z_1 and z_2 respectively be the left and right edge coordinates of the boundary (water surface) between the water block and the air.
Obtain a horizontal component (x component) of the force, f_x, which is exerted on the water block due to the pressure, per unit length along the y-axis, in terms of ρ, g, z_1, and z_2. Note that the atmospheric pressure P_0 exerts no net horizontal force on the water block. **0.8pt**

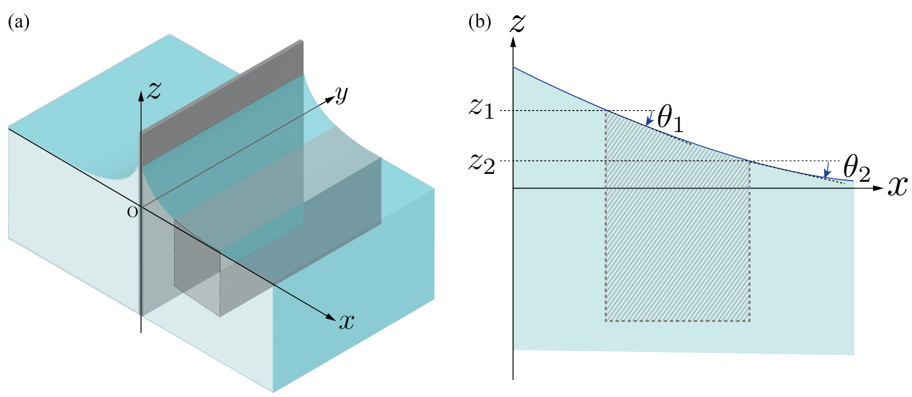


Fig. 3: Cutout form of water block on the water surface. (a) Bird's eye view and (b) cross-sectional view.

B.3	Surface tension acting on the water block is balanced with the force f_x discussed in B.2. We respectively define θ_1 and θ_2 as the angles between the water surface and the horizontal plane at the left and right edges. Express f_x in terms of γ, θ_1, and θ_2.	0.8pt

B.4	The following equation holds at an arbitrary point (x, z) on the water surface, $$\frac{1}{2}\left(\frac{z}{\ell}\right)^a + \cos\theta(x) = \text{constant}. \quad (1)$$ Determine the exponent a and express the constant ℓ in terms of γ and ρ. Note that this equation holds regardless of hydrophilic or hydrophobic board materials.	0.8pt

B.5	In Eq. (1) in B.4, we assume that variations of the water surface are slow and expand $\cos\theta(x)$ with respect to the derivative of $z(x)$, $z'(x)$ up to the second order. Then, differentiating the resultant equation with respect to x, we obtain the differential equation satisfied by $z(x)$. Solve this differential equation and determine $z(x)$ for $x \geq 0$ in terms of $\tan\theta_0$ and ℓ.	1.5pt

Part C. Interaction between two rods (3.5 points)

The identical rods A and B made of the same material floating in parallel on the water surface are placed at the same distance away from the y-axis (Fig.4).

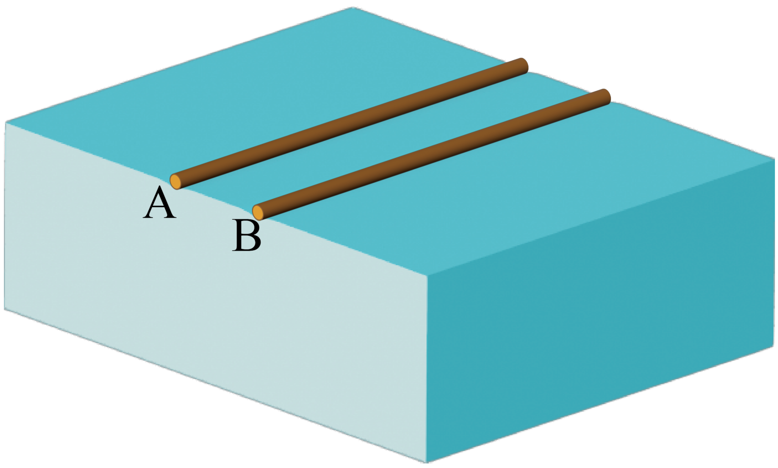


Fig. 4: Two rods A and B floating on the water surface.

C.1	As shown in Fig.5, we define the z-coordinates z_a and $z_b$as the contact positions between the rod B and the water surface, and the angles θ_a and θ_b. Determine the horizontal force component, F_x, on the rod B per unit length along the y-axis in terms of θ_a, θ_b, z_a, z_b, ρ, g, and γ.	1.0pt

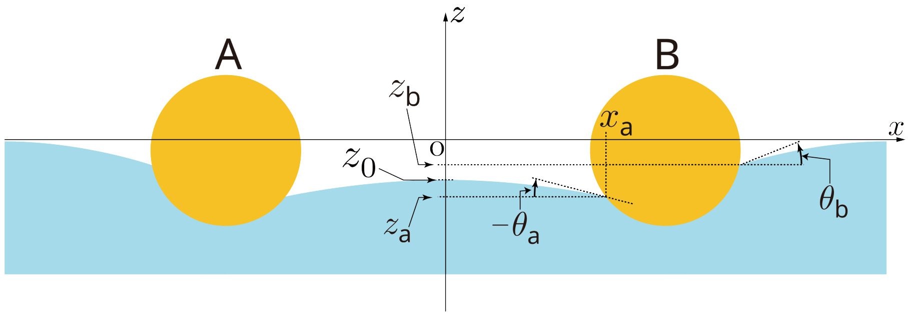


Fig. 5: Vertical cross-sectional view of two rods floating on the water surface.

C.2	We define the z-coordinate of the water surface, z_0, at the midpoint of two rods in the xz plane. Express the force F_x obtained in C.1 without using θ_a, θ_b, z_a, and z_b.	1.5pt

C.3	Let x_a be the x-coordinate of the contact point between the water surface and the left end of the rod B. Using the differential equation obtained in B.4, express the water level coordinate z_0 of the midpoint of these two rods A and B in terms of x_a and z_a. You can use the constant ℓ introduced in B.4.	1.0pt