# Solution / marking scheme – Characterizing Soil Colloids (10 points)

General rules
- In the following, “coefficients” refer to the numerical factors and do not include parameters.

## Part A. Analysis of motions of colloidal particles (1.6 points)

**A.1** (total 0.8 pt)

( 0.4 pt)

$$v_0 = \frac{I_0}{M}$$

partial points

(0.2 pt) $Mv_0 = I_0$ (A.1.1)

( 0.4 pt)

$$\tau = \frac{M}{\gamma}$$

- 0.4pt if the answers are $v_0 = M/\gamma$ and $\tau = I_0/M$.

partial points

(0.2 pt) $M\dot{v} = -\gamma v(t)$ (A.1.2)

**A.2** (total 0.8 pt)

( 0.6 pt)

$$v(t) = \sum_i \frac{I_i}{M} e^{-(t-t_i)/\tau}$$

- 0.4pt if $\frac{I_i}{M} e^{-(t-t_i)/\tau}$ is written. The subscript can be any dummy variable used in the summation symbol.
- 0.2pt if sum is taken (if $\Sigma$ is written).
- the range of sum is not considered here (even if it is wrong).
- $\tau = M/\gamma$ can be substituted.

( 0.2 pt)

the inequality specifying the range of $t_i$ that needs to be considered:
$0 < t_i < t$

- $<$ can be $\leq$ (full mark is given).
- 0.2pt (full mark) is given to $t_i < t$ (without $0 <$)
- No point is given to $t_i > 0$ solely.

## Part B. Effective equation of motion (1.8 points)

**B.1** (total 1.0 pt)

( 0.5 pt) Usable letters: $C, \delta, t$

$$\langle \Delta x(t) \rangle = 0$$

( 0.5 pt) Usable letters: $C, \delta, t$

$$\langle \Delta x(t)^2 \rangle = C\delta t$$

partial points

(0.3 pt) $$\Delta x(t) = \sum_{n=1}^{N} v_n \delta \tag{B.1.1}$$

- 0.2pt if $\delta$ is missing.

(0.2 pt) $$\langle \Delta x(t)^2 \rangle = \sum_{n=1}^{N} C\delta^2 = NC\delta^2 = C\delta t \tag{B.1.2}$$

- 0.2pt only if $C\delta t$ is written. 0.1pt if only $\sum_{n=1}^{N} C\delta^2$ or $NC\delta^2$ is written.

**B.2** (total 0.8 pt)

( 0.4 pt)

$$\alpha = -1$$

( 0.4 pt)

$$\beta = 1$$

## Part C. Electrophoresis (2.7 points)

**C.1** (total 0.5 pt)

( 0.5 pt) Usable letters: $v, \delta, n(x_0), \frac{dn}{dx}(x_0)$

$$N_+(x_0) = \frac{1}{2}n(x_0)v - \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta$$

- 0.3pt if $\delta$ or $A$ or both are multiplied unnecessarily (subtraction of 0.2pt)
- 0.4pt if either coefficient (or both) is wrong (subtraction of 0.1pt)
- 0.4pt if the sign of the second term is wrong (subtraction of 0.1pt)
- If more than one of the above mistakes are made, points to subtract accumulate.

partial points

(0.3 pt) $$N_+(x_0) = \int_{x_0 - v\delta}^{x_0} \frac{n(x)}{2\delta} dx \quad \text{or} \quad N_+(x_0) = \frac{v}{2}n(x_0 - v\delta/2) \tag{C.1.1}$$

- 0.2pt if $\delta$ or $A$ or both are multiplied unnecessarily (subtraction of 0.1pt)
- 0.2pt if any coefficient is wrong (subtraction of 0.1pt)
- 0.2pt if the integration range is $\int_{x_0}^{x_0+v\delta}$ (subtraction of 0.1pt)
- 0.2pt if $N_+(x_0) = \frac{v}{2}n(x_0 + v\delta/2)$ (subtraction of 0.1pt)
- If more than one of the above mistakes are made, points to subtract accumulate.

**C.2** (total 0.7 pt)

( 0.4 pt) Usable letters: $C, \delta, n(x_0), \frac{dn}{dx}(x_0)$

$$J_D(x) = -\frac{1}{2}\frac{dn}{dx}(x)C\delta$$

- 0.3pt if the sign or the coefficient is wrong (but pay attention to carryover from C.1).

partial points

(0.1 pt) $$N_-(x_0) = \frac{1}{2}n(x_0)v + \frac{1}{4}\frac{dn}{dx}(x_0)v^2\delta \tag{C.2.1}$$

( 0.1 pt) Usable letters: $C, \delta$

$$D = \frac{1}{2}C\delta$$

( 0.2 pt) Usable letters: $D, t$

$$\langle \Delta x(t)^2 \rangle = 2Dt$$

- No point if the answer includes $C$ or $\delta$.

**C.3** (total 0.5 pt)

( 0.5 pt) Usable letters: $n(x), T, Q, E, k$

$$\frac{dn}{dx} = \frac{n(x)}{kT} QE$$

partial points

$$\text{(0.3 pt)} \quad \Pi(x)A + n(x)A\Delta x QE = \Pi(x + \Delta x)A \tag{C.3.1}$$

**C.4** (total 0.5 pt)

( 0.3 pt)

$$\langle v(t) \rangle = \frac{QE}{\gamma}(1 - e^{-t/\tau})$$

- $\tau = M/\gamma$ can be substituted.

partial points

$$\text{(0.3 pt)} \quad M\frac{d\langle v(t) \rangle}{dt} = -\gamma \langle v(t) \rangle + QE \tag{C.4.1}$$

( 0.2 pt)

$$u = \frac{QE}{\gamma}$$

**C.5** (total 0.5 pt)

( 0.5 pt) Usable letters: $k, \gamma, T$

$$D = \frac{kT}{\gamma}$$

partial points

$$\text{(0.2 pt)} \quad J_D(x) = -\frac{DQE}{kT} n(x) \tag{C.5.1}$$

$$\text{(0.2 pt)} \quad J_Q(x) = \frac{QE}{\gamma} n(x) \tag{C.5.2}$$

## Part D. Mean square displacement (2.4 points)

**D.1** (total 1.0 pt)

( 1.0 pt)

$N_A = 5.6 \times 10^{23}\ \mathrm{mol}^{-1}$

- No reduction if the unit is missing.
- 0.8pt if the second digit is wrong but the value is in the range $5.5$–$5.7 \times 10^{23}$.

partial points

$$(0.5\ \mathrm{pt}) \quad \langle \Delta x^2 \rangle = \frac{RT\Delta t}{3\pi a\eta N_A} \tag{D.1.1}$$

- 0.3pt if both the answer of C.2 ($\langle \Delta x^2 \rangle = 2D\Delta t$) and that of C.5 ($D = \frac{kT}{\gamma}$) are given in the worksheet for D.1. The combination of them ($\langle \Delta x^2 \rangle = \frac{2kT\Delta t}{\gamma}$) is also acceptable. $k = R/N_A$ and $\gamma = 6\pi a\eta$ can be substituted here.
- No reduction if $t$ is used for $\Delta t$.

$$(0.3\ \mathrm{pt}) \quad \langle \Delta x^2 \rangle = 6.34\ \mu\mathrm{m}^2 \tag{D.1.2}$$

- No reduction if the value is in the range $6.2$–$6.4\ \mu\mathrm{m}^2$.
- 0.2pt if the value is in the range $4$–$9\ \mu\mathrm{m}^2$ or if the standard deviation of $\Delta x$ is in the range $2$–$3\ \mu\mathrm{m}$.
- Subtract 0.1pt if the unit is missing or wrong.

**D.2** (total 0.8 pt)

( 0.2 pt) Usable letters: $u, D, t$

$$\langle \Delta x^2 \rangle = (ut)^2 + 2Dt$$

( 0.2 pt)

$$\langle \Delta x^2 \rangle \propto \begin{cases} t & \text{for small } t \\ t^2 & \text{for large } t \end{cases}$$

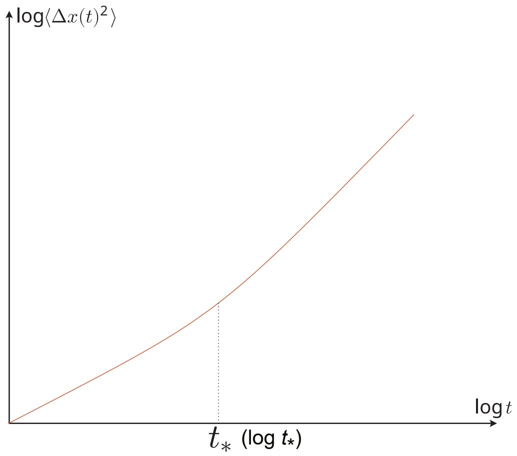
- 0.1pt independently for each answer.

( 0.2 pt)

$$t_* = \frac{2D}{u^2}$$

( 0.2 pt)

Points are given according to the criteria given below.




- 0.1pt if the graph is monotonically increasing and convex (no points if there are multiple curves that look like the answered graph)
- 0.1pt if $t_*$ is written between the two power-law regions (the label can be either $t_*$ or $\log t_*$).

**D.3** (total 0.6 pt)

( 0.6 pt)

$$\langle \Delta x^2 \rangle = \begin{cases} 2Dt & \text{for small } t \\ u_0^2 t^2 & \text{for intermediate } t \\ (u_0^2 \delta) t & \text{for large } t \end{cases}$$

- 0.2pt independently for each answer.
- Wrong answer in B.1 is not considered.

## Part E. Water purification (1.5 points)

**E.1** (total 1.5 pt)

( 1.5 pt)

$$c = \frac{8B^2\epsilon^3(kT)^5}{e^4 N_A A^2 q^6}$$

- 1.3pt if only the coefficient is wrong ($e$ is a part of the coefficient) (then no further partial point is given)

partial points

$$\text{(0.5 pt)} \quad \min U'(d) = 0 \tag{E.1.1}$$

- No point for $U'(d) = 0$ solely (without indicating what $d$ to consider) or $U'(a) = 0$.
- 0.2pt if the graph of the potential with an energy barrier (the graph first increases monotonically, then decreases monotonically) is drawn (this is the potential for $c < c_*$)
- independently, 0.2pt if the graph of the potential without an energy barrier (the graph increases monotonically) is drawn (this is the potential for $c > c_*$)

$$\text{(0.2 pt)} \quad U'(d) = \frac{A}{d^2} - \frac{B\epsilon(kT)^2}{q^2\lambda} e^{-d/\lambda} = 0 \tag{E.1.2}$$

$$\text{(0.2 pt)} \quad U''(d) = -\frac{2A}{d^3} + \frac{B\epsilon(kT)^2}{q^2\lambda^2} e^{-d/\lambda} = 0 \tag{E.1.3}$$

- 0.2pt (out of the 0.4pt right above) if both $U'(d) = 0$ and $U''(d) = 0$ are written as simultaneous equations, without their correct explicit forms.

$$\text{(0.2 pt)} \quad d = 2\lambda = \sqrt{\frac{Aq^2\lambda}{B\epsilon(kT)^2}} \tag{E.1.4}$$

$$\text{(0.3 pt)} \quad \lambda = \frac{e^2 A q^2}{4B\epsilon(kT)^2} \tag{E.1.5}$$

- 1.4pt is given in total if (E.1.5) is written.
- 1.2pt if only the coefficient is wrong ($e$ is a part of the coefficient)

**E.1 (cont.)**

Another solution: it is also physically reasonable to consider $\max U(d) = 0$ instead of (E.1.1), though this does not meet the requirements given in the question. Therefore, partial points may be given as follows if the question is answered along this line.

partial points

$$(0.5\text{ pt})\quad \max U(d) = 0 \tag{E.1.6}$$

- No point for $U(d) = 0$ solely (without indicating what $d$ to consider) or $U(a) = 0$.
- 0.2pt if the graph of the potential with an energy barrier that is higher than $U = 0$ or $U(d \to \infty)$ is drawn (this is the potential for $c < c_*$)
- independently, 0.2pt if the graph of the potential with an energy barrier that is lower than $U = 0$ or $U(d \to \infty)$ is drawn (this is the potential for $c > c_*$)

$$U(d) = -\frac{A}{d} + \frac{B\epsilon(kT)^2}{q^2}e^{-d/\lambda} = 0 \tag{E.1.7}$$

$$(0.2\text{ pt})\quad U'(d) = \frac{A}{d^2} - \frac{B\epsilon(kT)^2}{q^2\lambda}e^{-d/\lambda} = 0 \tag{E.1.8}$$

- No point for (E.1.7)
- 0.2pt if both $U(d) = 0$ are $U'(d) = 0$ are written as simultaneous equations

$$(0.5\text{ pt})\quad d = \lambda = \frac{eAq^2}{B\epsilon(kT)^2} \tag{E.1.9}$$

- 1.2pt is given in total if (E.1.9) is written.
- 1.0pt if only the coefficient is wrong ($e$ is a part of the coefficient)

$$(0.1\text{ pt})\quad c = \frac{B^2\epsilon^3(kT)^5}{2e^2N_AA^2q^6} \tag{E.1.10}$$

- 1.3pt is given in total if (E.1.10) is written.
- 1.1pt if only the coefficient is wrong ($e$ is a part of the coefficient)

# Solution / marking scheme – Neutron Stars (10 points)

General rules
- In the following, “coefficients” refer to the numerical factors and do not include parameters.

## Part A. Mass and stability of nuclei (2.5 points)

**A.1** (total 0.9 pt)

( 0.9 pt)

$A = 50$

- No reduction if $A = 5.0 \times 10^1$.
- 0.8 pt if the value is in the range 49.5—50.4.

partial points

$$(0.2\text{ pt})\quad \frac{B}{A} = a_V - a_S A^{-1/3} - \frac{a_C}{4} A^{2/3} \tag{A.1.1}$$

- No reduction if the difference from (A.1.1) is only the overall coefficient. This rule is applied throughout.

$$(0.1\text{ pt})\quad \frac{d(B/A)}{dA} = 0 \tag{A.1.2}$$

$$(0.2\text{ pt})\quad \frac{a_S}{3} A^{-4/3} - \frac{a_C}{6} A^{-1/3} = 0 \tag{A.1.3}$$

- Points for (A.1.2) are given if (A.1.3) is stated although (A.1.2) is not explicitly written.

$$(0.2\text{ pt})\quad A = \frac{2a_S}{a_C} \tag{A.1.4}$$

- 0.7 pt is given if the correct expression for $A$ appears even if the intermediate steps are not fully written.

**A.2** (total 0.9 pt)

( 0.9 pt)

$Z^* = 79$

- No reduction if $Z^* = 78$.
- 0.8 pt if the value is in the range 77.5–79.4.

partial points

(0.3 pt) $$-2a_C \frac{Z^*}{A^{1/3}} - 4a_{\text{sym}} \frac{2Z^* - A}{A} = 0 \tag{A.2.1}$$

(0.4 pt) $$Z^* = \frac{1}{1 + \dfrac{a_C}{4a_{\text{sym}}} A^{2/3}} \cdot \frac{A}{2} \tag{A.2.2}$$

- No reduction if $a_C/4a_{\text{sym}}$ is replaced by the numerical value in the range 0.007–0.008.

**A.3** (total 0.7 pt)

( 0.7 pt)

$C_{\text{fission}} = 0.70$

- No reduction if $C_{\text{fission}} = 0.7$.

partial points

(0.3 pt) $$a_S \left[ A^{2/3} - 2\left(\frac{A}{2}\right)^{2/3} \right] + a_C \left[ \frac{Z^2}{A^{1/3}} - 2\frac{(Z/2)^2}{(A/2)^{1/3}} \right] > 0 \tag{A.3.1}$$

- No point if $a_V$ is not canceled.

(0.2 pt) $$\frac{Z^2}{A} > \frac{2^{1/3} - 1}{1 - 2^{-2/3}} \cdot \frac{a_S}{a_C} \tag{A.3.2}$$

- Points for (A.3.1) are given if (A.3.2) is stated although (A.3.1) is not explicitly written.
- The coefficient may have different expressions, e.g., with $x = 2^{1/3}$,

$$\frac{x-1}{1-x^{-2}} = \frac{x^2}{1+x} = \frac{x}{1+x^{-1}} = \cdots = 0.702414\ldots$$

## Part B. Neutron star as a gigantic nucleus (1.5 points)

**B.1** (total 1.5 pt)

( 0.8 pt)

$a_{\text{grav}} = 6 \times 10^{-37}$ MeV

- No reduction if the unit is not written.
- 0.7 pt if only the order of magnitude is correct.

partial points

(0.4 pt) $$a_{\text{grav}} = \frac{3}{5}\frac{Gm_N^2}{R_0} \tag{B.1.1}$$

(0.2 pt) $$a_{\text{grav}} = \frac{3}{5}\frac{\hbar c m_N^2}{R_0 M_P^2} \tag{B.1.2}$$

- Points for (B.1.1) are given if (B.1.2) is stated although (B.1.1) is not explicitly written.
- No reduction if $\hbar$ is mistyped.

( 0.7 pt)

$A_c = 4 \times 10^{55}$

- No reduction for $A_c = 5 \times 10^{55}$.
- 0.6 pt if only the order of magnitude is correct.

partial points

(0.2 pt) $$a_V A - a_{\text{sym}} A + a_{\text{grav}} A^{5/3} > 0 \tag{B.1.3}$$

(0.3 pt) $$A_c = \left(\frac{a_{\text{sym}} - a_V}{a_{\text{grav}}}\right)^{3/2} \tag{B.1.4}$$

- Points for (B.1.3) are given if (B.1.4) is stated although (B.1.3) is not explicitly written.

## Part C. Neutron star in a binary system (6.0 points)

**C.1** (total 1.0 pt)

( 1.0 pt)

$$\Delta\tau_{\mathrm{II}} = \left(1 - \frac{\Delta\phi}{c^2}\right)\Delta\tau_{\mathrm{I}}$$

- No points if the coefficient is wrong.

partial points

(0.3 pt) $v^2 = 2g\Delta h = 2\Delta\phi$ or $v = \sqrt{2\Delta\phi}$ (C.1.1)

(0.5 pt) $\Delta\tau_{\mathrm{II}} = \sqrt{1 - v^2/c^2}\,\Delta\tau_{\mathrm{I}}$ or $\Delta\tau_{\mathrm{II}} = \sqrt{1 - 2\dfrac{\Delta\phi}{c^2}}\,\Delta\tau_{\mathrm{I}}$ (C.1.2)

- Points for (C.1.1) are given if (C.1.2) is stated although (C.1.1) is not explicitly written.

**C.2** (total 1.8 pt)

( 1.8 pt)

$$\Delta t = \frac{2GM_{\mathrm{WD}}}{c^3}\log\left(\frac{4|x_N|x_E}{d^2}\right)$$

- No reduction if 4 is missing in log.
- No reduction if $|x_N|$ is written as $-x_N$.
- 0.1 pt is subtracted if the modulus in $|x_N|$ is missing.
- No points if other coefficients are wrong.

partial points

(0.5 pt) $t_{\text{E-N}} = \displaystyle\int_{x_N}^{x_E} \frac{dx}{c_{\mathrm{eff}}(x)}$ or $\Delta t_{\text{E-N}} = \dfrac{\Delta x}{c_{\mathrm{eff}}(x)}$ (C.2.1)

(0.4 pt) $t_{\text{E-N}} \simeq \dfrac{1}{c}\displaystyle\int_{x_N}^{x_E} dx\left(1 + \frac{2GM_{\mathrm{WD}}}{c^2\sqrt{x^2+d^2}}\right)$ (C.2.2)

- 0.1 pt is subtracted if the coefficient is wrong.

(0.3 pt) $\Delta t = \dfrac{2GM_{\mathrm{WD}}}{c^3}\displaystyle\int_{x_N}^{x_E} \frac{dx}{\sqrt{x^2+d^2}}$ (C.2.3)

(0.3 pt) Inside the logarithm: $\sqrt{x_N^2+d^2} + x_N \simeq \dfrac{d^2}{2|x_N|}$ and $\sqrt{x_E^2+d^2} - x_E \simeq \dfrac{d^2}{2x_E}$ (C.2.4)

**C.3** (total 1.8 pt)

( 1.8 pt)

$$\Delta t_{\text{max}} - \Delta t_{\text{min}} = \frac{2GM_{\text{WD}}}{c^3} \log(4/\varepsilon^2)$$

- No reduction if log is written as ln.

partial points

(0.6 pt) $\Delta t_{\text{max}} = \frac{2GM_{\text{WD}}}{c^3} \log(4x_E/L\varepsilon^2)$ (C.3.1)

- No subtraction points if the factor in log is different but consistent with that in C.2.
- 0.1 pt is subtracted if the coefficient is wrong.

(0.2 pt) Because of $x_N > 0$ the approx. in log is changed: $x_N + \sqrt{x_N^2 + d^2} \simeq 2L$ (C.3.2)

(0.4 pt) $\Delta t_{\text{min}} = \frac{2GM_{\text{WD}}}{c^3} \ln(x_E/L)$ (C.3.3)

- Points for (C.3.2) are given if (C.3.3) is stated although (C.3.2) is not explicitly written.
- 0.1 pt is subtracted if the coefficient is wrong.

(0.3 pt) Points are given if $L$ and $x_E$ dependence is canceled in log. (C.3.4)

**C.4** (total 0.8 pt)

( 0.8 pt)

$$M_{\text{WD}}/M_\odot = 0.5$$

- No reduction if the value is in the range 0.4–0.5.

partial points

(0.2 pt) $\varepsilon^2 \simeq 2 \times (1 - 0.99989) = 0.00022$ (C.4.1)

(0.2 pt) From the given graph, $\Delta t_{\text{max}} - \Delta t_{\text{min}} \approx 50\,\mu\text{s}$ (C.4.2)

- No reduction if the value from the graph is in the range 40–50 $\mu$s.

(0.2 pt) $M_{\text{WD}}/M_\odot \simeq 5/\ln(4/\varepsilon^2)$ (C.4.3)

- No reduction if the numerator is in the range 4–5.

**C.5** (total 0.4 pt)

( 0.4 pt)

$$p = -\frac{3}{2} \quad \text{or} \quad -1.5$$

- No points if the search sign is wrong.

partial points

(0.3 pt) $R^3\omega^2 = (\text{const.})$ (C.5.1)

**C.6** (total 0.2 pt)

( 0.2 pt)

The most appropriate profile is (b).

# Solution / marking scheme – Water and Objects (10 pt)

General rules
- In the following, "coefficients" refer to the numerical factors and do not include parameters.

## Part A. Merger of water drops (2.0 pt)

**A.1** (total 2.0 pt)

( 2.0 pt)

$v = 0.23$ m/s
- No deduction if the answer falls within the range $0.22$ m/s $\leq v \leq 0.24$ m/s

partial points

The surface energy per drop before the merger:

$$(0.4 \text{ pt}) \quad E = 4\pi a^2 \gamma \tag{A.1.1}$$

The surface energy difference:

$$(0.6 \text{ pt}) \quad \Delta E = 4\pi \left(2 - 2^{2/3}\right) a^2 \gamma \tag{A.1.2}$$

The transfer of surface energy to kinetic energy :

$$(0.4 \text{ pt}) \quad M v^2/2 = k \Delta E \tag{A.1.3}$$

where $M = 4\pi a^3 \rho/3 \times 2 = 8\pi a^3 \rho/3$ is the mass of the drop after the merger.
- No partial point will be given if the factor $k$ is missing.

Numerical evaluation:

$$v = \sqrt{\frac{2k\Delta E}{M}} = \sqrt{3\left(2 - 2^{2/3}\right)\frac{k\gamma}{\rho a}} = \sqrt{3\left(2 - 2^{2/3}\right) \times \frac{0.06 \times \left(7.27 \times 10^{-2}\right)}{\left(1.0 \times 10^{3}\right) \times \left(100 \times 10^{-6}\right)}} = 0.23\cancel{2} \text{ m/s}$$

## Part B. A vertically placed board (4.5 pt)

**B.1** (total 0.6 pt)
Usable letters: $\rho, g, z, P_0$

( 0.6 pt)

$$P = P_0 - \rho g z$$

- No point will be given for $P = P_0 + \rho g z$

Commentary

The expression, $P = P_0 - \rho g z$, holds for both $z < 0$ and $z > 0$, as long as $z$ is inside the water.

**B.2** (total 0.8 pt)
Usable letters: $\rho, g, z_1, z_2$

( 0.8 pt)

$$f_x = \frac{1}{2}\rho g\left(z_2^2 - z_1^2\right)$$

- Give 0.6 pt for $f_x = \rho g\left(z_2^2 - z_1^2\right)$
- Give 0.4 pt for $f_x = \frac{1}{2}\rho g\left(z_1^2 - z_2^2\right)$

Commentary

Because the atmospheric pressure $P_0$ exerts no net horizontal force on the water block, we have

$$f_x = \int_{z_2}^{z_1}(-\rho g z)dz = \frac{1}{2}\rho g\left(z_2^2 - z_1^2\right)$$

**B.3** (total 0.8 pt)
Usable letters: $\gamma, \theta_1, \theta_2$

( 0.8 pt)

$$f_x = \gamma\cos\theta_1 - \gamma\cos\theta_2$$

- Give 0.6 pt for $f_x = \gamma\cos\theta_2 - \gamma\cos\theta_1$
- Give 0.4 pt for $f_x = \gamma\cos\theta_2 + \gamma\cos\theta_1$ or $f_x = -\gamma\cos\theta_2 - \gamma\cos\theta_1$.

**B.4** (total 0.8 pt)

( 0.4 pt)

$$a = 2$$

- No point will be given for $a \neq 2$.

Usable letters: $\gamma, \rho$

( 0.4 pt)

$$\ell = \sqrt{\frac{\gamma}{\rho g}}$$

- If an unnecessary coefficient is included as a factor, 0.2 pt will be deducted.

**B.5** (total 1.5 pt)
Usable letters: $\tan\theta_0, \ell$

( 1.5 pt)

$$z(x) = -\ell \tan\theta_0 e^{-x/\ell}$$

- Deduct 0.2 pt for $z(x) = -\ell \sin\theta_0 e^{-x/\ell}$ or $z(x) = -\ell\theta_0 e^{-x/\ell}$.

partial points

$z' = \tan\theta$ leads to

$$(0.2\text{ pt})\quad \cos\theta = \frac{1}{\sqrt{1+(z')^2}} \tag{B.5.1}$$

$$(0.1\text{ pt})\quad \cos\theta \simeq 1 - \frac{1}{2}(z')^2 \tag{B.5.2}$$

Plug this into Eq.(1) to obtain,

$$(0.2\text{ pt})\quad \frac{z^2}{\ell^2} - z'^2 = \text{const.} \tag{B.5.3}$$

Take the derivative of both sides with respect to $x$ :

$$(0.5\text{ pt})\quad z'' = \frac{z}{\ell^2} \tag{B.5.4}$$

which is the differential equation which determines the water surface form.
General solution:

$$(0.2\text{ pt})\quad z = Ae^{x/\ell} + Be^{-x/\ell} \tag{B.5.5}$$

The boundary condition, $z(\infty) = 0$, leads to

$$(0.1\text{ pt})\quad A = 0 \tag{B.5.6}$$

The boundary condition, $z'(0) = \tan\theta_0$, leads to

$$(0.2\text{ pt})\quad B = -\ell\tan\theta_0 \tag{B.5.7}$$

## Part C. Interaction between two rods (3.5 pt)

**C.1** (total 1.0 pt)
Usable letters: $\theta_{\mathrm{a}}, \theta_{\mathrm{b}}, z_{\mathrm{a}}, z_{\mathrm{b}}, \rho, g, \gamma$

( 1.0 pt)

$$F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\left(\cos\theta_{\mathrm{b}} - \cos\theta_{\mathrm{a}}\right)$$

- Give 0.8 pt for $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\left(\cos\theta_{\mathrm{a}} - \cos\theta_{\mathrm{b}}\right)$
- Give 0.6 pt for $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) + \gamma\cos\theta_2 + \gamma\cos\theta_1$ or $F_x = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) - \gamma\cos\theta_2 - \gamma\cos\theta_1$.

partial points

The holizontal component of the force due to the pressure is

$$\text{(0.6 pt)} \quad \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z)\, dz = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right) \qquad \text{(C.1.1)}$$

Commentary

Comment 1: How to apply the experience in B.1 is as follows. Let $z_{\mathrm{bottom}}$ the $z$-coordinate at the bottom of the rod, then from the discussion in B1, we see

$$F_x = \int_{z_{\mathrm{bottom}}}^{z_{\mathrm{a}}} (-\rho g z) dz + \left(-\int_{z_{\mathrm{bottom}}}^{z_{\mathrm{b}}} (-\rho g z) dz\right) = \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z) dz$$

Comment 2: The fact that the contribution due to the pressure does not depend on the shape of the cross-section can be demonstrated as follows. The pressure at the point $s$ on the contour $C$ along the cross-sectional boundary is

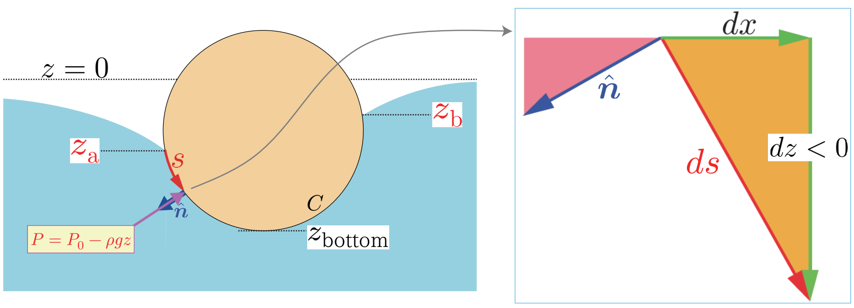
$$-P\hat{n}ds = (-P_0 + \rho g)\hat{n}ds.$$

Let $\hat{x}$ the unit vector pointing the positive $x$-direction and noting $\hat{x}\cdot\hat{n}ds = dz$ (see the figure shown below), the holizontal component becoms and its holizontal component becomes

$$-P\hat{n}\cdot\hat{x}ds = -P_0 dz + \rho g dz.$$

Integrating along the contour $C$, we obtain

$$\oint_C (-P\hat{n}\cdot\hat{x}ds) = \int_{z_{\mathrm{a}}}^{z_{\mathrm{b}}} (\rho g z) dz = \frac{1}{2}\rho g\left(z_{\mathrm{b}}^2 - z_{\mathrm{a}}^2\right)$$



**C.2** (total 1.5 pt)
Unusable letters: $\theta_\mathrm{a}, \theta_\mathrm{b}, z_\mathrm{a}, z_\mathrm{b}$

( 1.5 pt)

$$F_x = -\frac{1}{2}\rho g z_0^2$$

- Give 1.3 pt for $F_x = -\rho g z_0^2$.
- Give 0.8 pt for $F_x = \frac{1}{2}\rho g z_0^2$.

partial points

Apply the boundary conditions to Eq. (1) to obtain

$$(0.6 \text{ pt}) \quad \underbrace{\frac{1}{2}\rho g z_\mathrm{a}^2 + \gamma\cos\theta_\mathrm{a}}_{x=x_\mathrm{a}} = \underbrace{\frac{1}{2}\rho g z_0^2 + \gamma}_{x=0} \tag{C.2.1}$$

- Give 0.4 pt for $\rho g z_\mathrm{a}^2 + \gamma\cos\theta_\mathrm{a} = \rho g z_0^2 + \gamma$

$$(0.6 \text{ pt}) \quad \underbrace{\frac{1}{2}\rho g z_\mathrm{b}^2 + \gamma\cos\theta_\mathrm{b}}_{x=x_\mathrm{b}} = \underbrace{\gamma}_{x\to\infty} \tag{C.2.2}$$

- Give 0.4 pt for $\rho g z_\mathrm{b}^2 + \gamma\cos\theta_\mathrm{b} = \rho g z_0^2$

$F_x$ is obtained by subtracting (C2.1) from (C2.2).

**C.3** (total 1.0 pt)
Usable letters: $x_\mathrm{a}, z_\mathrm{a}$

( 1.0 pt)

$$z_0 = \frac{2z_\mathrm{a}}{e^{x_\mathrm{a}/\ell} + e^{-x_\mathrm{a}/\ell}}$$

- Correct alternative answer: $z_0 = \dfrac{z_\mathrm{a}}{\cosh(x_\mathrm{a}/\ell)} = z_\mathrm{a}\mathrm{sech}(x_\mathrm{a}/\ell)$

partial points

General solution: $z(x) = Ae^{x/\ell} + Be^{-x/\ell}$
Taking into account the left-right symmetry, we obtain,

$$(0.3\ \mathrm{pt})\quad A = B \tag{C.3.1}$$

Boundary condition, $z(0) = z_0$ leads to

$$(0.3\ \mathrm{pt})\quad A + B = z_0 \tag{C.3.2}$$

Find the coefficients:

$$(0.2\ \mathrm{pt})\quad A = z_0/2 \tag{C.3.3}$$

$$(0.2\ \mathrm{pt})\quad B = z_0/2 \tag{C.3.4}$$