1 Three balls

Let the ball A is given an initial velocity v along the axis Y, which is perpendicular to the rod. The total momentum of the system conserves, therefore the center-of-mass (CM) of the system moves with a constant velocity:

$$v_{\text{CM}} = \frac{mv}{3m} = \frac{v}{3}$$

along Y. In what follows, we will work in the CM frame of reference, which is an inertial system of reference. Therefore, in the CM frame the laws of conservation of energy, momentum, and the angular momentum hold true. The initial velocities of the three balls along Y are:

$$v_A = \frac{2v}{3}, \qquad v_B = v_C = -\frac{v}{3}.$$

Correspondingly, the total kinetic energy of the balls is:

$$E = \frac{mv^2}{2}\left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{mv^2}{3}$$

and the total angular momentum with respect to the CM equals:

$$L = m\frac{2v}{3}\ell - m\frac{v}{3}(-\ell) = mv\ell.$$

In any moment the three balls form an isosceles triangle with an angle 2φ at the top vertex. The distance between A and C is minimal when either $\varphi = 0$, or $\dot{\varphi} = 0$. For $\varphi = 0$, however, the laws of conservation are not consistent with the rigidity of the rods. Therefore, at the minimal distance $\dot{\varphi} = 0$, and in this particular instance the system behaves as a rigid body whose moment of inertia with respect to the CM can be obtained through:

$$I = \frac{L^2}{2E} = \frac{3}{2}ml^2. \tag{1}$$

On the other hand, the moment of inertia I could be found independently from geometric considerations. Although, I could be found using the distances from the balls to the CM (medicentre of a triangle), it is more convenient to use this relatively unknown formula for the moment of inertia of a collection of point masses with respect to CM:

$$I = \frac{\sum_{i,j} m_i m_j (\mathbf{r}_i - \mathbf{r}_j)^2}{\sum_i m_i}.$$

In our case:

$$I = \frac{m^2}{3m}\left(AB^2 + BC^2 + AC^2\right) = \frac{m}{3}\left(2\ell^2 + d^2\right). \tag{2}$$

From (1) and (2), we obtain the minimal distance:

$$d = \ell\sqrt{\frac{5}{2}} \approx 1.58\,\ell$$

2 Solenoid

Magnetic field gives rise to magnetization of water molecules, i.e. each of the molecules becomes a magnetic dipole. Inhomogeneous magnetic field exerts a force to a magnetic dipole. Water molecules being diamagnetic are pushed away from the region of stronger magnetic field. Water reaches a state with mechanical equilibrium where the magnetic force is balanced by a force due to pressure gradient. Hence, in mechanical equilibrium, regions with stronger magnetic field correspond to smaller pressures. When the current in the solenoid is increased, the pressure differences grow, and at a certain moment, a region will appear where the pressure is smaller than the saturation pressure of the water vapour. This is the moment when water starts boiling.

To begin with, let us discuss possible reasonable approximations. First, we can neglect the water column pressure which is only 2 to 4 percents of the atmospheric pressure. Second, the saturation pressure of water vapour under normal conditions is also much smaller than the atmospheric one and therefore can be neglected. Thus we can say that boiling starts when the pressure drop due to magnetic field becomes equal to p_0.

So, we need to relate the pressure difference caused by the magnetic field to the magnetic field strength. Notice that if a region with magnetic field B is filled with water, the magnetic field energy density is $B^2/(2\mu_r\mu_0)$; meanwhile, if there is no water, the energy density is $B^2/(2\mu_0)$. So, we can ascribe the energy density difference

$$\Delta w = (\mu_r^{-1} - 1)B^2/(2\mu_0)$$

to the interaction of water and magnetic field. Next, consider the following thought experiment. We push away a small volume V of water from the neighbourhood of a point P in the water where the magnetic field strength is B; the displaced water fills in a narrow layer at the top of the water surface of equal volume. Assuming that the magnetic field is negligibly small at the top, by comparing the initial and final states, we conclude that the total interaction energy is reduced by $V(\mu_r^{-1} - 1)B^2/(2\mu_0)$. When pushing away water from P we perform mechanical work pV, where p is the pressure at point P. At the upper surface of the water, the moving interface performs mechanical work p_0V so that the net mechanical work performed by water during this process is $V(p_0 - p)$. Due to energy conservation law, $V(\mu_r^{-1} - 1)B^2/(2\mu_0) = V(p_0 - p)$ so that

$$p_0 = p + (\mu_r^{-1} - 1)B^2/(2\mu_0).$$

Note that the way how we derived this relationship is completely analogous to how the Bernoulli law is derived, and in fact, the obtained equality can be interpreted as a modified Bernoulli law for zero speed where the volume density of potential energy in gravity field ρgh is replaced with $(\mu_r^{-1} - 1)B^2/(2\mu_0)$ — the energy density of magnetic interaction. This equality can be simplified by noting that $\mu_r^{-1} - 1 = -\chi/\mu_r \approx -\chi$ so that

$$p_0 - p = -\chi B^2/(2\mu_0).$$

As discussed above, the boiling condition is $p \approx 0$, hence

$$B = \sqrt{-2\mu_0 p_0/\chi}.$$

Finally, we apply the formula for magnetic field strength inside a long solenoid $B = \mu_0 IN/\ell$ to find

$$I = \frac{\ell}{N}\sqrt{\frac{2p_0}{-\chi\mu_0}} = 4.4\,\text{kA}.$$

3 Staircase

A Since $n = -y/h = (x/\lambda)^{2/3}$, $x(n) = n^{2/3}\lambda$. The distance between the steps is

$$d_n = x(n+1) - x(n) \approx \frac{\mathrm{d}x(n)}{\mathrm{d}n} = \frac{2}{3}\,\lambda\, n^{-1/3} = n^{-1/3} \cdot 30\,\mu\mathrm{m}.$$

B Equilibrium energy value, being minimum, must be stable against small perturbations of the crystal shape. Allowed are perturbations which conserve the total volume of the crystal. In other words a small horizontal displacement of one step must be accompanied by an equal and opposite displacement of another step.

The energy change $\epsilon_n(\delta)$ associated with a small horizontal displacement δ of the n-th step is

$$\epsilon_n(\delta) = \mu\Big((d_n+\delta)^\nu - d_n^\nu + (d_{n+1}-\delta)^\nu - d_{n+1}^\nu\Big) \approx \\ \approx \mu\nu\left(d_n^{\nu-1} - d_{n+1}^{\nu-1}\right)\delta.$$

In order for $\epsilon_n(\delta) + \epsilon_m(-\delta)$ to be zero for arbitrary n and m it is necessary to require that the factor in the parentheses does not depend on n:

$$d_n^{\nu-1} - d_{n+1}^{\nu-1} = \text{const}.$$

Substituting $d_n \propto n^{-1/3}$, we get[1]:

$$n^{(1-\nu)/3} - (n+1)^{(1-\nu)/3} \approx \frac{1-\nu}{3} n^{(1-\nu)/3-1} = \text{const},$$

$$\frac{1-\nu}{3} - 1 = 0 \implies \nu = -2.$$

The interaction energy corresponds to that of two dipoles in 2D:

$$E(d) \propto \frac{1}{d^2}.$$

[1]Trivial solutions $\nu = 0$ and $\nu = 1$ imply that the total energy within given constraints does not depend on the shape of the crystal.

Part A.

Turn the CO_2 sensor on. It takes 2-3 minutes for the sensor self-calibration, after that the measurement starts.

Before the main experiment concentration of CO_2 in the vessel is equal to the concentration c_0 of CO_2 in the atmosphere, let us determine it

$$c_0 = (0.050 \pm 0{,}005)\%.$$

Now let us consider the diffusion process theoretically and express typical diffusion time τ via the membrane geometrical parameters.

The change of CO_2 molecules number inside the vessel per unit time is equal

$$\frac{d}{dt}(cV) = -jpS_0$$

where pS_0 is the net area of channels. The flow through the channels is

$$j = D\frac{c - c_0}{h}$$

and we get differential equation for concentration $c(t)$

$$\frac{dc}{dt} = -\frac{DpS_0}{Vh}(c - c_0)$$

The solution is

$$c(t) = c_0 + C\exp\left(-\frac{t}{\tau}\right)$$

where

$$\tau = \frac{Vh}{pS_0D}$$

According to the problem text the Knudsen flow takes place in the membrane channels. In this case the diffusion coefficient D is determined by the collisions of molecules with the walls of the channel

$$D = \frac{1}{3}vd.$$

Knowing the molar mass of carbon dioxide $\mu = 44\frac{g}{mol}$, we calculate the thermal velocity of CO_2 molecules for room temperature $T = 295$ K:

$$v = \sqrt{\frac{8RT}{\pi\mu}} = 376\ m/s$$

Finally we have

$$\tau = \frac{3Vh}{pS_0vd}$$

Calculate the full membrane area $S_0 = \frac{\pi}{4} d_w^2 = 1.33\ sm^2$. We measure the length and the inner diameter of the vessel:

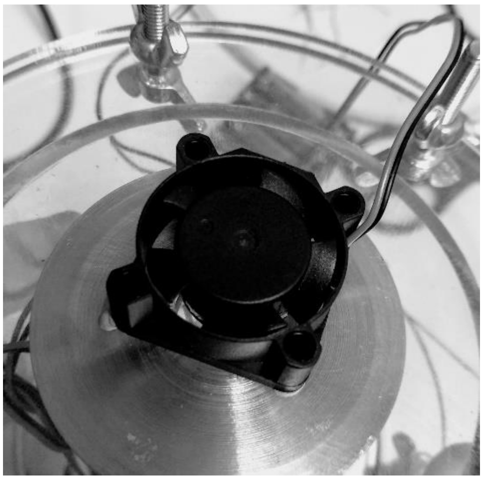
$$L = 5.0\ sm$$

$$D_{in} = 7.4\ sm$$

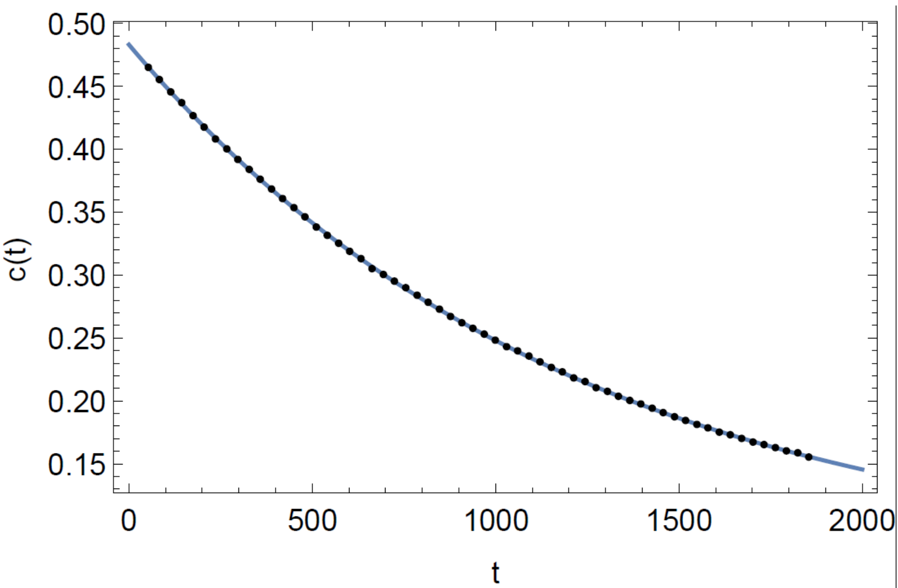
and calculate its inner volume $V = 215\ sm^3$. The volume of fan, sensor and tubes way be neglected.

Now let's procced with the experiment. We will use our own lungs as a source of carbon dioxide. Let's take a breath and exhale the air into the cylinder through the tube, then close both tubes with clips. A second tube is needed in order for the air to better circulate in the vessel during the exhalation.

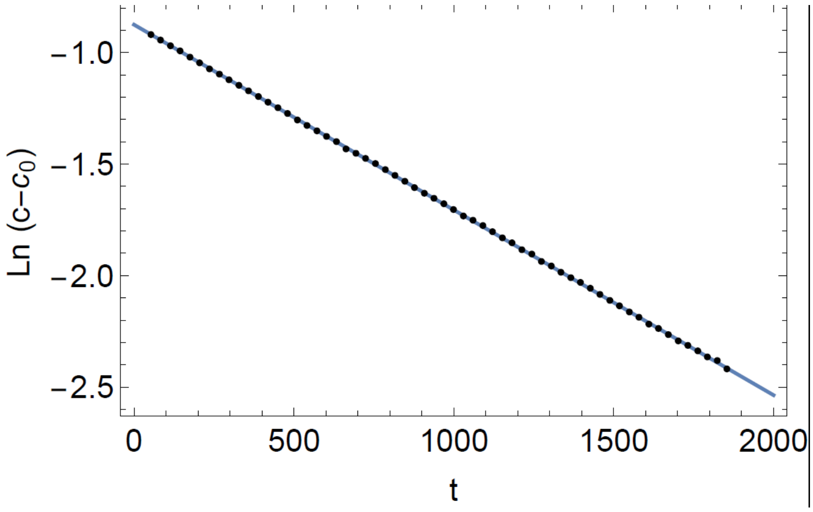
To ensure that the concentration of CO_2 in the vessel is the same at each moment, turn on the internal fan, it mixes the air in the vessel. To ensure that the concentration of CO_2 outside the membrane is equal to c_0, turn on the second fan to blow on the membrane outside.



Measure the dependence $c(t)$,where the concentration is measured in %, and the time in seconds. The results are shown on the plot.



Let's plot a linearized graph in the coordinates $t, Ln\ (c - c_0)$.



From its slope we calculate the required time

$$\tau = 1204 \pm 10\ sec$$

Slope and its uncertainty were calculated with OLS.

Notes.

1. The CO_2 concentration in the exhaled air is 4% and the sensor's working limit is 0.5%. Therefore, if you blow into the installation strongly, the sensor is off scale. Participants can wait until the CO_2 concentration in the vessel drops below 0.5% and then start the measurements. There are also two other ways to reduce the CO_2 concentration in the vessel. You can unscrew the cover of the installation (with tubes, not with the sample!) and utilizing the fan blow the installation with atmospheric air. Another way is to open both tubes and suck the air out through one of them. The atmospheric air comes inside the vessel through the second tube. This method is the fastest and easiest.

2. The equipment setups for parts B,C are independent on equipment for part A, so the optical measurements can be performed simultaneously with the diffusional. Moreover, the sensor stores the data.

3. If the participant does not use the second (external) fan, the measured value τ will be about 2 times larger, which will give incorrect answer for the pore diameter.

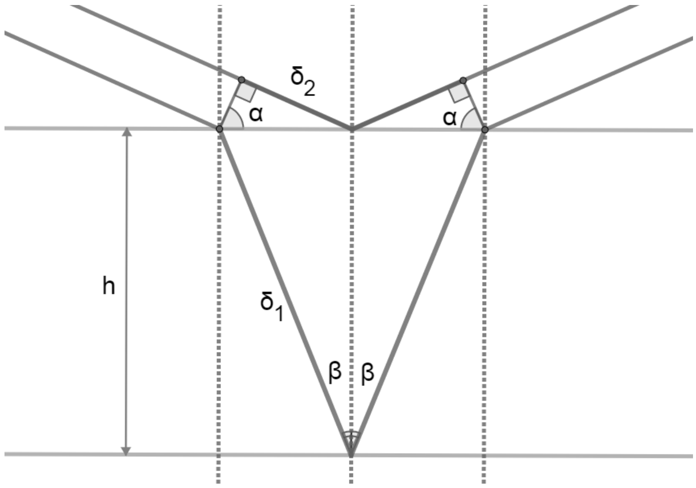
Part B.

Optical path difference between beams reflected from top and bottom sides of the membrane:

$$\delta = 2\delta_1 - 2\delta_2 + \frac{\lambda}{2}$$

$\frac{\lambda}{2}$ appears because of phase change in case of reflection from the surface with higher refractive index.

$$\frac{\sin\alpha}{\sin\beta} = n$$



$$\delta_1 = n\frac{h}{\cos\beta}$$

$$\delta_2 = \frac{h}{\cos\beta}\sin\beta\sin\alpha$$

$$\delta - \frac{\lambda}{2} = \frac{2h}{\cos\beta}(n - \sin\beta\sin\alpha) = \frac{2h}{\cos\beta}(n - n\sin^2\beta) = \frac{2nh}{\cos\beta}\cos^2\beta = 2nh\cos\beta = 2nh\sqrt{1-\sin^2\beta}$$

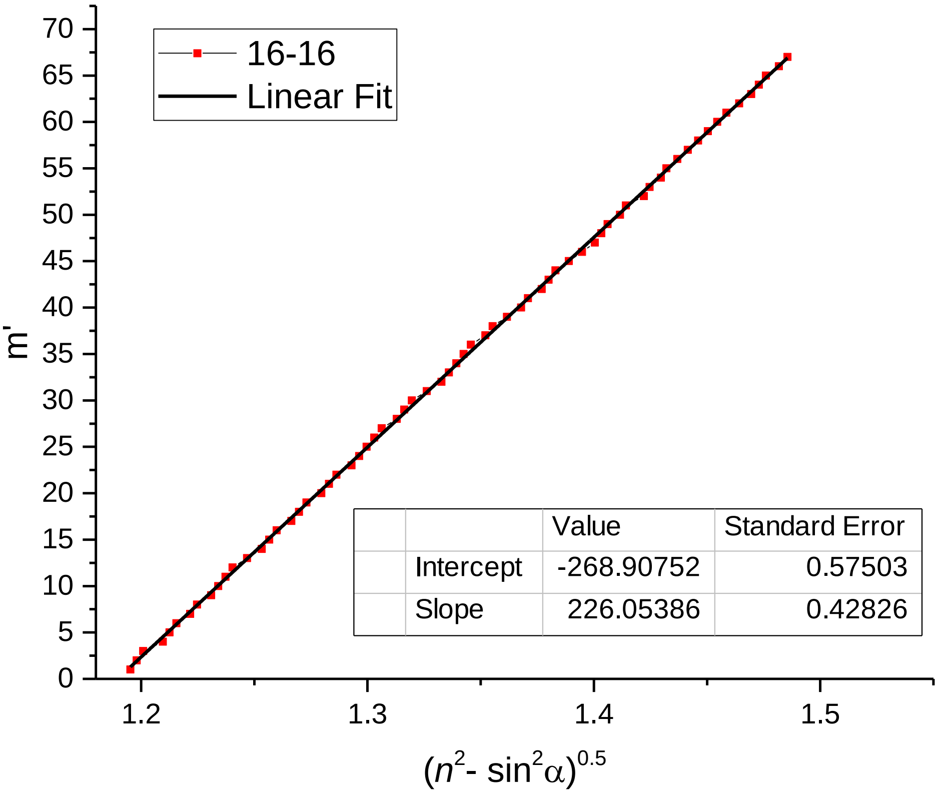
$$= 2nh\sqrt{1 - \frac{\sin^2\alpha}{n^2}} = 2h\sqrt{n^2 - \sin^2\alpha}$$

$$\delta = 2h\sqrt{n^2 - \sin^2\alpha} + \frac{\lambda}{2}$$

(1)

Reflectance maximums can be observed if $\delta = m\lambda$, where m is integer. Reflectance maximums can be observed if $\delta = (m + 1/2\,)\lambda$.

Rotating the sample, we can see dozens of minimums and maximums. $\sqrt{n^2 - \sin^2\alpha}$ depending on the number of minimum is linear according to the equation (1), as shown on the graph. Slope is equal $2h/\lambda$.

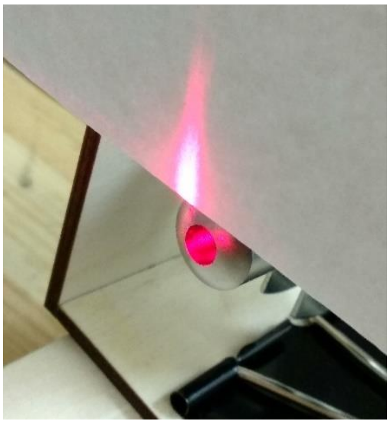


Slope and its uncertainty were calculated with OLS.

$$h = 74.6 \pm 0.2\ \mu m$$

This data is shown for demonstration. During the competition, students were not supposed to measure positions of that much number of minimums and plot a graph, because this does not improve accuracy. The best solution is to rotate the sample and calculate number of minimums between two angles α. The more is number of minimums N, the better is accuracy. In this case, uncertainty ε_h may be estimated as $\varepsilon_h = 1/N$.

The sample is not fixed on the rotating table. Before the measurement, one should set a zero using the reflected laser beam.



Part C.

In calculations me denote $\Delta n^* \equiv n_2 - n_1$, $\Delta\beta \equiv \beta_1 - \beta_2$.

From Snell's law

$$n_1 \sin\beta_1 = n_2 \sin\beta_2$$

Optical path between beams is $\delta = h(n_1 \cos\beta_1 - n_2 \cos\beta_2)$. Taking condition $n_2 - n_1 \ll n_1$ into account, we should derive δ in terms of n and Δn^*.

$$\delta = h(n_1 \cos\beta_1 - n_2 \cos\beta_2) = h\left(n_1 \cos\beta_1 - n_1 \frac{\sin\beta_1}{\sin\beta_2}\cos\beta_2\right) = n_1 h \sin\beta_1 \left(\frac{\cos\beta_1}{\sin\beta_1} - \frac{\cos\beta_2}{\sin\beta_2}\right)$$
$$= nh \sin\beta\,(\cot\beta_1 - \cot\beta_2)$$

$$\frac{\cos\beta_1 - \cos\beta_2}{\beta_1 - \beta_2} \simeq (\cot\beta)' = -\frac{1}{\sin^2\beta}$$

$$\delta = -\frac{nh}{\sin\beta}\Delta\beta$$

(C1)

Now we will derive $\Delta\beta$ in terms of Δn^*.

$$\frac{n_1}{n_2} = \frac{\sin\beta_1}{\sin\beta_2}$$

$$\frac{n_2 - \Delta n^*}{n_2} = \frac{\sin(\beta_2 + \Delta\beta)}{\sin\beta_2} \simeq \frac{\sin\beta_2 + \cos\beta_2\,\Delta\beta}{\sin\beta_2}$$

$$\Delta\beta = -\frac{\Delta n^*}{n}\frac{\sin\beta}{\cos\beta}$$

(C2)

Combining (C1) and (C2), we obtain the equation:

$$\delta = \frac{h}{\cos\beta}\Delta n^*$$

(C3)

Relation between Δn and Δn^* may be found from formulae given in the task:

$$\frac{1}{n_2^2} = \frac{\cos^2\beta_2}{n_0^2} + \frac{\sin^2\beta_2}{n_e^2}$$

(C4)

$$\frac{1}{n_1^2} = \frac{1}{n_0^2}$$

(C5)

(C4)-(C5):

$$\frac{1}{n_2^2} - \frac{1}{n_1^2} = \frac{\cos^2\beta_2}{n_0^2} - \frac{1}{n_0^2} + \frac{\sin^2\beta_2}{n_e^2}$$

$$\frac{n_1^2 - n_2^2}{n_1^2 n_2^2} = \sin^2\beta_2 \frac{n_0^2 - n_e^2}{n_0^2 n_e^2}$$

$$\frac{n_1 + n_2}{n_1^2 n_2^2} \Delta n^* = \frac{n_0 + n_e}{n_0^2 n_e^2} \sin^2 \beta_2 \, \Delta n$$

$$\Delta n^* = \sin^2 \beta \, \Delta n$$

(C6)

Taking account of (C3), it yields:

$$\delta = \frac{h}{\cos \beta} \sin^2 \beta \, \Delta n$$

(C7)

Polarized waves may interfere in projections. We should use 2 polarizers with polarization axes pointed at $45°$ with membrane rotation axis.

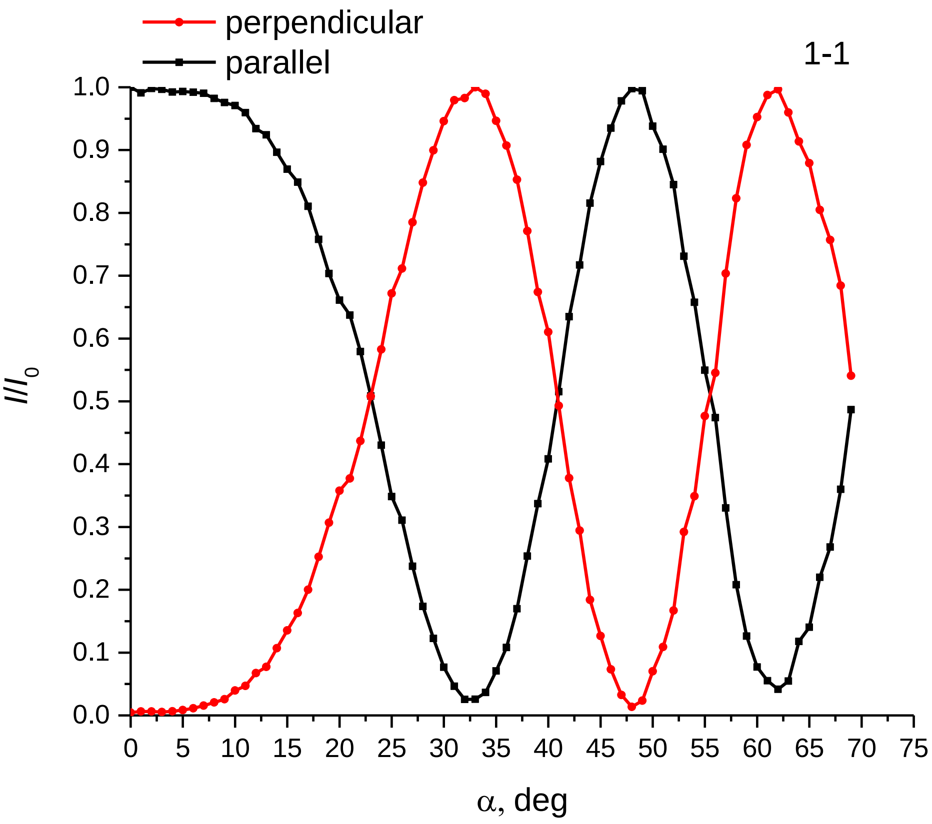
First polarizer let beams 1 and 2 have equal intensity. Second polarizer projects waves on its polarization axis, so they can interfere. Polarization axes of polarizers may be perpendicular or parallel. Using the described setup, one can observe minimums with zero intensity and bright maximums.

Over setups me be used to observe the main effect. One can arrange the setup without second polarizer (laser beam is polarized itself), or place polarizers at angles differs from $45°$.

In any case, when we rotate the membrane, we can find three transmittance extrema. If we use the setup described above, optical paths in these points will be: $\delta_1 = \lambda/2$, $\delta_2 = \lambda$, $\delta_3 = 3/2\lambda$.

Transmittance depending on the angle of incidence is shown on the graph. To obtain this perfect data, we also measured transmittance without the second polarizer to take account of reflectance rising at big angles.

Students were not supposed to take a lot of measurements or plot the graph, this isn't necessary to complete the task. To calculate Δn with best possible accuracy, we just need to determine angles of three transmittance extrema by any way.



Δn values are calculated according to the equation (C7).

δ	$\alpha,^\circ$	$\beta,^\circ$	Δn
$1/2\,\lambda$	32.5	20.1	0.0352
λ	48	27.2	0.0375
$3/2\,\lambda$	62	31.8	0.0405

$$\varepsilon_{\Delta n} = \frac{1}{\Delta n}\sqrt{\sum_{i}^{N}\frac{(\Delta n_{\mathrm{i}} - \Delta n)^2}{N(N-1)}}$$

$$\Delta n = 0.0378 \pm 0.0015$$

Using the plot $\Delta n(p)$, we determine the porosity of the sample.

$$p = 13.5 \pm 0.5\%$$

Coda

In this part, we can determine diameter of pores, using results, obtained in previous parts:

$$\tau = 1204 \pm 10$$

$$h = 74.6 \pm 0.2 \text{ мкм}$$

$$p = 13.5 \pm 0.5\%$$

In part A we deriver the equation

$$\tau = \frac{3Vh}{pS_0vd}$$

It yields

$$d = \frac{3Vh}{pS_0\, v\tau}$$

Where v is thermal velocity of CO_2 ($T \simeq 300K$, more accuracy isn't needed):

$$v = \sqrt{\frac{8RT}{\pi\mu}} = 376\, m/s$$

V is volume of the cylindrical vessel, it can be measured ruler:

$$V = \pi \frac{D_{in}^2}{4} L$$

$$L = 5.0\, cm$$

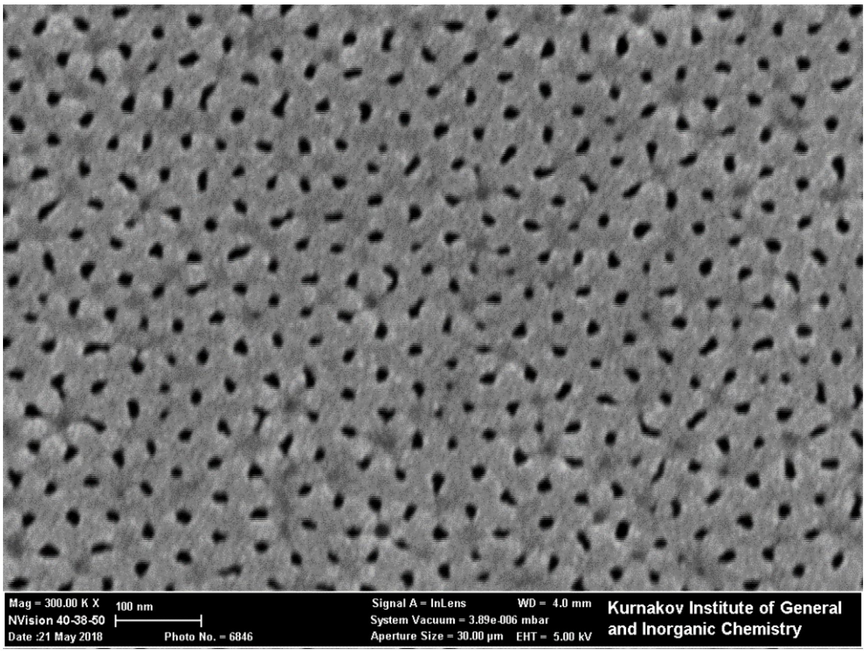
$$D_{in} = 7.4\, cm$$

$$V = 215 cm^3$$

Numerical answer is:

$$d = 6.0 \pm 0.5\, nm$$

Here you can see the surface image of the membrane studied in this experimental problem. This image was obtained with scanning electron microscope (SEM). Average diameter of channels is about 15 nm. The reason of difference between these experimental results is that diffusion inside pores is not the only limiting stage even with two fans are turned on. Moreover, some fraction of channels is branched and terminated (A.A. Noyan et al. / Electrochimica Acta 226 (2017) 60–68).



This problem was developed and designed by Alexey Noyan, Alexander Kiselev and Fedor Tsybrov. Samples of anodic aluminum oxide were fabricated in MSU by Kirill Napolskii, Alexey Leontiev, Ilya Roslyakov and Sergey Kushnir.

If you have any questions about this experimental problem, please don't hesitate to contact the author *noyan@phystech.edu.*