T1: A Leak

Let p_1, V_1, T_1 denote the (time dependent) pressure, volume, and temperature in the upper chamber, and p_2, V_2, T_2 — those in the lower one. Note that $V_1 \equiv V$ does not change.

Consider a parcel of volume v below the diaphragm containing n moles of helium. It is convenient to imagine it bounded by two fictitious free thin massless pistons. During slow perturbations the parcel undergoes an adiabatic process. The pressure and the temperature for the parcel are actually the pressure and the temperature for entire lower chamber p_2 and T_2. The energy conservation for the parcel is

$$0 = p_2 \mathrm{d}v + \mathrm{d}\left(\frac{3}{2} n_v R T_2\right) = \frac{5}{2} p_2 \mathrm{d}v + \frac{3}{2} v \mathrm{d}p_2.$$

This gives

$$v^5 p_2^3 = \text{const}, \qquad (1)$$

$$T_2^5 p_2^{-2} = \text{const}\,. \qquad (2)$$

The leak begins when the pressure below the diaphragm exceeds that in the upper chamber by $\Delta p \equiv p_0 - p = mgH/V = px$, where

$$x = \frac{mgH}{pV}. \qquad \textbf{(a., b., c.)}$$

a. We may let $v = V_2$ before that.

$$V^5 p^3 = V_0^5 (p + \Delta p)^3 = V_0^5 p^3 (1+x)^3$$

$$V_0 = V(1+x)^{-3/5} \qquad \textbf{(a.)}$$

b. The energy conservation for the whole system:

$$0 = p_1 \mathrm{d}V_1 + \mathrm{d}\left(\frac{3}{2} n_1 R T_1\right) + p_2 \mathrm{d}V_2 + \mathrm{d}\left(\frac{3}{2} n_2 R T_2\right) = \frac{5}{2}\left(p_1 \mathrm{d}V_1 + p_2 \mathrm{d}V_2\right) + \frac{3}{2}\left(V_1 \mathrm{d}p_1 + V_2 \mathrm{d}p_2\right) = \frac{5}{2} p_2 \mathrm{d}(V + V_2) + \frac{3}{2}(V + V_2)\mathrm{d}p_2,$$

since the pressure above the diaphragm remains lower than that below by the same margin Δp during the later process and $\mathrm{d}V_2 = \mathrm{d}(V + V_2)$. Similarly to (1), we get

$$(V + V_2)^5 p_2^3 = \text{const}\,.$$

The pressure p_2' in the lower chamber when the piston touches the diaphragm is found from the equation

$$V^5 p_2'^3 = (V + V_0)^5 p_0^3 = V^5 \left(1 + \frac{1}{(1+x)^{3/5}}\right)^5 p^3 (1+x)^3,$$

$$p_2' = p\left(1 + (1+x)^{3/5}\right)^{5/3}. \qquad (3)$$

The pressure in the upper chamber at this moment is

$$p_1' = p_2' - \Delta p = p\left(\left(1 + (1+x)^{3/5}\right)^{5/3} - x\right).$$

The temperature in the upper chamber is found from the equations of state $pV = nRT$ and $p'V = (2n)RT'$

$$T_1' = \frac{T}{2}\left(\left(1 + (1+x)^{3/5}\right)^{5/3} - x\right). \qquad \textbf{(b.)}$$

The temperature and the pressure in the lower chamber are related by (2). Substituting (3) we get

$$T_2' = T\left(\frac{p_2'}{p}\right)^{2/5} = T\left(1 + (1+x)^{3/5}\right)^{2/3}. \qquad \textbf{(c.)}$$

Preliminary grading scheme

a1	It's stated (or written as a formula) that the process is adiabatic	0.5
a2	Relation between V and p is found in adiabaic process	1.0
a3	Condition on when the diaphragm leaks	0.5
a4	Answer for V_0	1.0
b1	Energy conservation for the whole system in differential form. If conservation law is written only for one half but heat transfer is taken into account: 0.5 pts.	1.0
b2	Internal energy for a mono-atomic gas	0.5
b3	Usage of $V_1 = \text{const}$	0.2
b3	Usage of $p_2 - p_1 = \text{const}$	0.3
b4	Relation between V_2 and p_2	1.0
b5	Equation to find p_2' (or T_2') before the end	1.0
b6	Usage of $n' = 2n$	0.5
b7	Answer for T_1'	0.5
c1	Relation between T_1' and T_2'	1.0
c2	Answer for T_2'	1.0

Arithmetic or typo errors gives half of point (rounded up to 0.1) for the item and is not considered as a mistake afterwards.

T2: Thread on cylinder

Fig. 1 shows the cylinder and the loop from three different angles; P denotes the pulled point of the loop, while O is the top point of the thread.

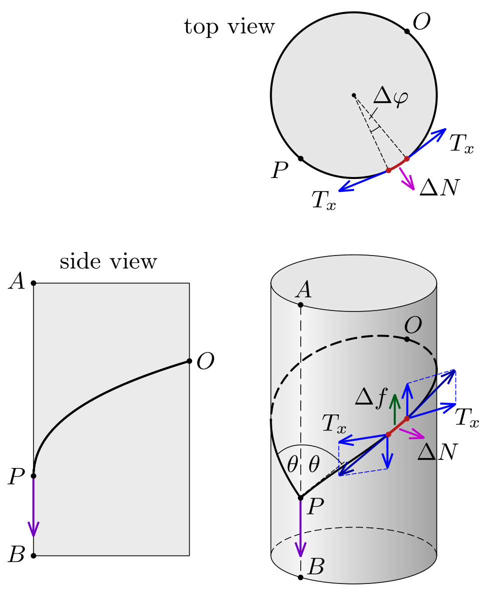


Fig. 1

Imagine that the side of the cylinder is cut along the generatrix AB passing through point P, and then the side (including the loop) is unfolded as shown in *Fig. 2*. In this figure points A and A', B and B', P and P' are equivalent, respectively. Let us introduce a Cartesian coordinate system on this unfolded plane so that point O is the origin, axis z is parallel with the axis of cylinder and directed downwards, axis x is perpendicular to z (i.e. horizontal).

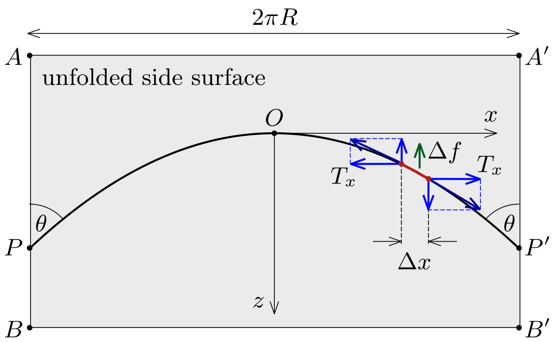


Fig. 2

Consider the forces acting on a small piece of the thread (with horizontal projection Δx) indicated with red line in both figures. These are the tensions at both ends of the small piece exerted by neighbouring parts of the thread, the normal force ΔN and the friction force Δf exerted by the cylinder. On the verge of slipping, the direction of Δf is parallel to the z-axis. Since the small piece of thread is in equilibrium, the x-component of the tension is the same everywhere:

$$T_x = \text{const.}$$

The normal force ΔN can be determined by looking at the top view of the loop in *Fig. 1*. The polar angle corresponding to small piece of thread is $\Delta\varphi = \Delta x/R$, so the force balance in the radial direction can be written as

$$2T_x \underbrace{\sin\frac{\Delta\varphi}{2}}_{\approx\Delta\varphi/2} - \Delta N = 0 \quad\longrightarrow\quad \Delta N = T_x\frac{\Delta x}{R}\,. \tag{4}$$

The frictional force on the verge of slipping is given by

$$\Delta f = \mu\Delta N\,. \tag{5}$$

Thus, the force balance on the small piece of thread in the z direction (see *Fig. 2*):

$$T_x\left.\frac{dz}{dx}\right|_{x+\Delta x} - T_x\left.\frac{dz}{dx}\right|_{x} - \Delta f = 0\,, \tag{6}$$

where we expressed the z-component of the tension forces with T_x and the tangent dz/dx. Using the three equations above and taking the limit $\Delta x \to 0$, we get the differential equation

$$\frac{d^2z}{dx^2} = \frac{\mu}{R}\,,$$

where we used the fact that $T_x \neq 0$. By direct integration and taking into account the boundary conditions $z(0) = 0$ and $z'(0) = 0$ we get

$$z(x) = \frac{\mu}{2R}x^2\,,$$

so the shape of the thread on the unfolded side surface (*Fig. 2*) can be described by a parabola.

The thread needs to span over the entire cylinder, so its length can be calculated as

$$L_0 = \int_{-\pi R}^{\pi R}\sqrt{dx^2 + dz^2} = 2\int_0^{\pi R}\sqrt{1+\left(\frac{dz}{dx}\right)^2}\,dx\,.$$

Substituting the $z(x)$ function we get:

$$L_0 = 2\int_0^{\pi R}\sqrt{1+\left(\frac{\mu x}{R}\right)^2}\,dx = \left(\frac{R}{\mu}\right)2\int_0^{\pi\mu}\sqrt{1+\xi^2}\,d\xi\,,$$

where we introduced the notation $\xi = \mu x/R$. Using the integral given in the text of the problem:

$$L_0 = \pi R\sqrt{1+(\pi\mu)^2} + \frac{R}{\mu}\,\text{arcsinh}(\pi\mu)\,.$$

If the length of the thread is shorter than the length calculated here, then there is no solution satisfying the thread length constraint, i.e. the thread cannot slip.

Note. In the limit $\mu \to 0$ the loop slips even if $L = 2\pi R$, which should be reproduced by our final formula. Using the relation for the inverse hyperbolic function given in the problem text, then expanding the logarithm in Taylor series in linear order around 0 we get:

$$\text{arcsinh}\,x \equiv \ln\left(x+\sqrt{1+x^2}\right) \approx \ln(x+1) \approx x\,,$$

so for small values of μ we get

$$L_0 \approx \pi R + \frac{R}{\mu}(\pi\mu) = 2\pi R\,.$$

Grading scheme: T2	
2-i. A figure or figures reflecting the correct geometry such as: the loop is a non-planar curve in all figures (0.5 p), existence of exactly one cusp (0.5 p, if the „back” is hidden, 0 p).	0.5 p + 0.5 p
2-ii. Realizing that on the verge of slipping the frictional force is parallel with the axis of the cylinder for every small piece of the thread and $\Delta f = \mu\Delta N$. (If any of the two is missing, then 0.2 p)	0.5 p
2-iii. Correct equation for the force balance in z direction involving frictional force or the load (which is pulling down the loop) and the z components of the tension.	0.5 p
2-iv. T_x = const. + correct explanation based on the balance in x direction. If the physics is incompatible with the geometrical assumptions (e.g. planar curve for the loop and the existence of a frictional force acting on small pieces), no points are given	0.5 p + 1.5 p
2-v. Expressing the normal force ΔN acting on a small segment with T_x and dx (or $d\phi$). If the relation $x = R\varphi$ or $dx = Rd\varphi$ is not used here or anywhere else, 1.5 p is given. If the expression is wrong but ΔN is proportional to the curvature ($1/R$), 1.0 p is given.	2.0 p

2-vi. Deriving the correct differential equation for $z(x)$ (if the diff. equation is wrong due to any reason, 0 p)	1.5 p
2-vii. Solving the diff. equation for $z(x)$ correctly (including boundary conditions). If only one integral is evaluated correctly, 0.3 p are given. For stating only the boundary conditions (both $z(0)$ and $z'(0)$) 0.2 p.	1.0 p
2-viii. Writing down the length of the thread in terms of an integral of $z(x)$, i.e. writing down the length constraint	0.5 p
2-ix. Evaluating the integral correctly (factor mistake in calculation 0.5 p, wrong units 0.2 p)	1.0 p
Total T2:	**10.0 p**

General guidelines for marking:

- Granularity for marks is 0.1 p.
- A simple numerical error resulting from a typo is punished by 0.2 p unless the grading scheme explicitly says otherwise.
- Errors which cause dimensionally wrong results are punished by at least 50 % of the marks unless the grading scheme explicitly says otherwise.
- Propagating errors are not punished repeatedly unless they either lead to considerable simplifications or wrong results whose validity can easily be checked later.

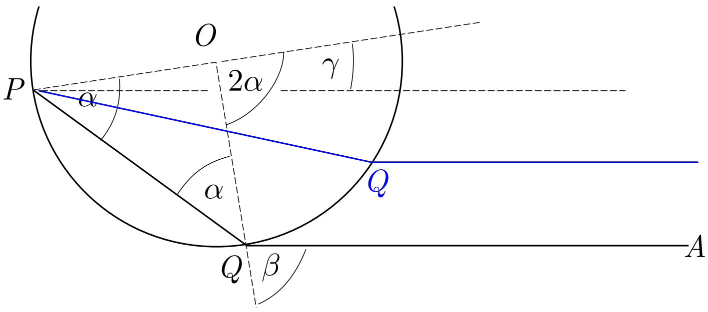
T3: Glass ball

To begin with, let us notice that if a ray coming from a point P on the stripe is refracted at point Q at the surface of the ball towards a very distant point A (which denotes the aperture of the camera lens), the ray will remain in the plane PQO where O is the centre of the sphere. This means that those rays which arrive from P to A must lay in the plane POA, and the rays can be conveniently depicted in the POA-plane.

The angle $\gamma = \gamma(\alpha)$ between vectors PO and QA is given by

$$\gamma \equiv 2\alpha - \beta \tag{7}$$
$$n \sin\alpha = \sin\beta \tag{8}$$

and is a non-monotonous function of α which achieves its maximum γ_0 by a certain α_0. This means that for a fixed P and A, if $\angle OPA < \gamma_0$, there are two such angles α and, hence, two such positions Q_1 and Q_2 for the point Q that the ray PQA will reach the point A (here we have assumed that the point A is at a very big distance). So, when viewed from the point A, the image of the point P is split into two points Q_1 and Q_2. On the other hand, if $\angle OPA > \gamma_0$, rays from P cannot reach A, and such points on the stripe cannot be seen in the photo. If $\angle OPA = \gamma_0$, the two images Q_1 and Q_2 merge into a single point bridging the two images of a piece of stripe into a closed loop.



Now it becomes clear that the place where images Q_1 and Q_2 merge into a single image Q_0 is the key to finding the coefficient of refraction. Indeed, the point Q_0 can be found in the photo as the point where the blue ellipse (the images of a segment of the stripe in red light as the blue shadow is where the red light is missing) is touching a radius of the ball, see the figure below (we need to find tangent point with the radius because we need to consider plane QOA which projects into a line through the ball's centre). There is no difference between taking the tangent to the outer edge of the elliptical stripe and taking the tangent to the inner edge of it (point Q_0' in the photo below). Then, $\sin\beta_0 = n\sin\alpha_0$ can be determined from the photo as the ratio of the lengths h and R, $h/R \approx 0.765$, where h denotes the distance of the ball's centre O from the line Q_0A, measurable in the photo as the length OQ_0 (when looking from the distant point A, we can see only the perpendicular-to-OA component of the segment OQ_0, and R is the ball's radius.

Since we look for an extremum of γ, upon taking differentials from Eqns. (7,8), we obtain

$$2\mathrm{d}\alpha = \mathrm{d}\beta$$
$$n\cos\alpha\,\mathrm{d}\alpha = \cos\beta\,\mathrm{d}\beta$$

from where

$$n\cos\alpha = 2\cos\beta. \tag{9}$$

$$n = \sqrt{\sin^2\beta + 4\cos^2\beta} = \sqrt{4 - 3\sin^2\beta} \approx 1.498 \approx 1.50.$$

In order to find Δn, we could find in a similar way n_V ($|OQ_0''| \approx 0.755R$, $n \approx 1.513 \approx 1.51$), but the result would have a huge relative uncertainty as $|OQ_0|$ and $|OQ_0''|$ have very similar lengths. A much more precise result will be obtained if we base our calculations on the segment length $|ST|$, see the figure below. Q_0'' is where the blue rays have an extremum for γ while Q_0' and Q_0'' are two red rays originating from the same point on the stripe. From the photo we can measure $|ST| \approx 0.20R$, hence we can use the small parameter $SQ_0''/R \approx 0.10$.

Rays from S, T, and Q_0'' arrive to the lens aperture, so all these rays (which originate from the same point P on the stripe) have the same value of γ. So, we have a set of equations

$$\gamma \equiv 2\alpha_R - \beta_R = 2\alpha_V - \beta_V \tag{10}$$
$$n_V \sin\alpha_V = \sin\beta_V \tag{11}$$
$$(n_V - \Delta n)\sin\alpha_R = \sin\beta_R \tag{12}$$
$$n_V\cos\alpha_V = 2\cos\beta_V, \tag{13}$$

and we would like to get an expression relating Δn to $\sin\beta_{V1} - \sin\beta_{V2}$, where the indices 1 and 2 relate to the two different solutions. Expressing $\alpha_R = \alpha_V + \delta$ we obtain from Eq. (10) that $\beta_R = \beta_V + 2\delta$. Now, if we expand Eq. (12) into Taylor series, neglect the smallest term with $\Delta n\delta$, and keep in mind Eq. (11), we obtain

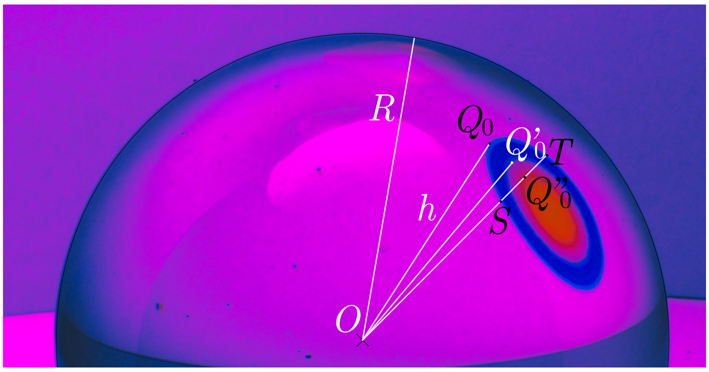
$$n_V\delta\cos\alpha_V - n_V\delta^2\frac{\sin\alpha_V}{2} - \Delta n\sin\alpha_V = 2\delta\cos\beta_V - 2\delta^2\sin\beta_V.$$

Here the two linear-in-δ terms cancel out due to Eq. (13) so that with $\sin\beta_V = n_V\sin\alpha_V$ we obtain

$$\Delta n = 3n_V\delta^2/2. \tag{14}$$

On the other hand, $|ST|/R = \sin\beta_{R1} - \sin\beta_{R2} \approx \sin(\beta_V + 2\delta) - \sin(\beta_V - 2\delta) = 4\delta\cos\beta_V$, hence

$$\Delta n = \frac{3n_V}{2}\left(\frac{|ST|}{4R\cos\beta_V}\right)^2 = \frac{3n_V}{32}\frac{|ST|^2}{R^2-h^2} \approx 0.0132 \approx 0.013. \tag{15}$$



Grading scheme

A-i

- Proving that we can see a loop because for a range of points on the thread, a single point of thread creates two images: 0.7 pts
- The endpoints of that thread create one image forming thereby a closed loop: 0.3 pts

Else if only the fact that rays coming to A must be in the PQO plane is stated: 0.3 pts

A-ii Drawing a ray diagram where we can see that two rays originating from a single point arrive both to the lens while crossing the ball's surface at different points and following the Snell's law: 1.0 pt
Else if such idea is demonstrated by words or by a rough sketch (which does not obey Snell's law): 0.5 pts

B-i Relating angle β to the measurable distance h in the photo (any point on the blue or red ellipse will earn the mark): 1pt

B-ii Measuring ratio of h to R: 0.5 pts

B-iii The idea of using point Q_0: 1 pt (using Q'_0 will earn only 0.5 pts as this corresponds to the violet light)

B-iv Obtaining a correct expression of γ as a function of α (or equivalent calculations in different parametrisation): 1 pt

B-v Finding extremum and hence, the final expression of n as a function of β or h: 1pt. If an incorrect expression is obtained from reasonable physics because of mistakes in algebraic manipulation: 0.5 pts

B-vi If the numerical answer is correct within ± 0.03 and is found using reasonable physics: 0.5 pts. Else if the numerical answer is correct within ± 0.1 and is found using reasonable physics: 0.3 pts. No point if the answer is guessed or is found using completely wrong or irrelevant physics.

C-i The idea of using the width or length of the blue ellipse: 1 pt (red ellipse cannot be used as we don't know if the center of the red ellipse corresponds to the edge of the thread, or to a point at its middle)

C-ii Obtaining Eq (14) or something equivalent based on considering neighbouring red and violet rays: 1 pt. If an incorrect expression is obtained from reasonable physics because of mistakes in algebraic manipulation: 0.5 pts

C-iii Expressing Δn correctly in terms of measurable quantities: 0.5 pts

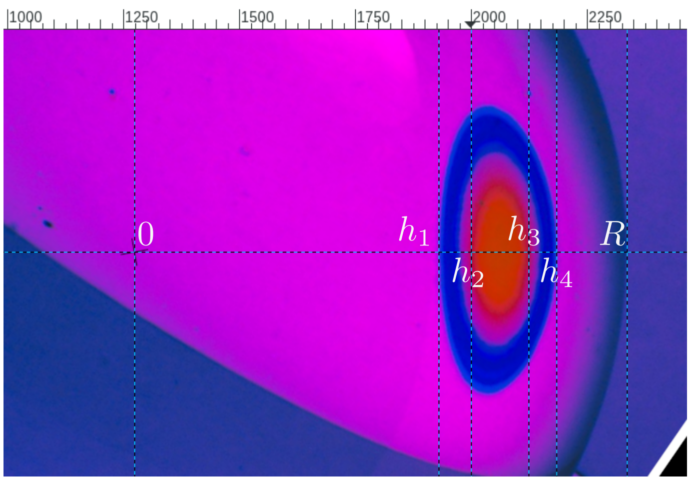
C-iv Finding numerical answer correct within ± 0.003: 0.5 pts. (0 pts if this answer is obtained without considering the width or height of the blue ellipse.)

Solution 2

One may alternatively prefer to consider a plane, perpendicular to the thread and containing the ball centre O and the camera A. This is the plane of symmetry for the system and corresponds to the maximum width of the ovals. The same arguments as in the first Solution lead to

$$\gamma = \beta - 2\arcsin\frac{\sin\beta}{n} = \arcsin\frac{h}{R} - 2\arcsin\frac{h}{nR}.$$

For $n < 2$ this function is not monotonous and may give the same result for two different values of h (that is why the mapping of an arc is a closed oval).



To get the refraction index for red we solve the equation

$$\arcsin\frac{h_1}{R} - 2\arcsin\frac{h_1}{n_R R} = \arcsin\frac{h_4}{R} - 2\arcsin\frac{h_4}{n_R R},$$

which gives $n_R \approx 1.51$.

One could repeat the previous argument to find n_V. One could also compare the images of the same thread elements in different colours. This gives 4 equations

$$\arcsin\frac{h_{1,4}}{R} - 2\arcsin\frac{h_{1,4}}{n_R R} = \arcsin\frac{h_{2,3}}{R} - 2\arcsin\frac{h_{2,3}}{n_V R},$$

where $h_{1,4}$ means we can replace it with the value of h_1 or h_4. Similar rule applies for $h_{2,3}$. From each equation, we can solve for one value of n_V and then take their average. This results in $n_V - n_R \approx 0.0145$.

Grading scheme

A-i

- Proving that we can see a loop because for a range of points on the thread, a single point of thread creates two images: 0.7 pts
- The endpoints of that thread create one image forming thereby a closed loop: 0.3 pts

Else if only the fact that rays coming to A must be in the PQO plane is stated: 0.3 pts

A-ii Drawing a ray diagram where we can see that two rays originating from a single point arrive both to the lens while crossing the ball's surface at different points and following the Snell's law: 1.0 pt
Else if such idea is demonstrated by words or by a rough sketch (which does not obey Snell's law): 0.5 pts

B-i Relating angle β to the measurable distance h in the photo (any point on the blue or red ellipse will earn the mark): 1pt

B-ii Measuring ratio of h to R: 0.5 pts

B-iii The idea of using points on the blue ellipse lying on the same radius of the sphere: 1 pt

B-iv Obtaining the correct equation to solve for n_R numerically: 2 pts. If an incorrect equation is obtained from reasonable physics because of mistakes in algebraic manipulation: 1 pt

B-v If the numerical answer is correct within ± 0.03 and is found using reasonable physics: 0.5 pts. Else if the numerical answer is correct within ± 0.1 and is found using reasonable physics: 0.3 pts. No point if the answer is guessed or is found using completely wrong or irrelevant physics.

C-i The idea of using points on the red (or both) ellipse(s) lying on the same radius of the sphere: 1 pt

C-ii Obtaining at least one correct equation to solve for n_V numerically: 1.5 pts. If an incorrect equation is obtained from reasonable physics because of mistakes in algebraic manipulation: 1 pt

C-iii Finding numerical answer correct within ± 0.003: 0.5 pts.

E1: Hidden wire

Theoretical background

As shown in Fig. 1, the horizontal projection $\vec{B}_\mathrm{h}$ of the magnetic induction $\vec{B}_\mathrm{w}$ of the wire has the same direction and is perpendicular to the wire in all points of the xy plane. It is clear that $\vec{B}_\mathrm{h}$ makes with North (y) direction an angle $\psi = 180^\circ - \theta$, where θ is the angle between the direction of the current and the positive x-direction. The magnetic needle points along the vector $\vec{B} = \vec{B}_\mathrm{h} + \vec{B}_\mathrm{E}$ of the total magnetic induction. As evident from the vector triangle on Fig. 1(a), the deflection angle φ can be obtained through the sine-theorem:

$$\frac{B_\mathrm{h}}{B_\mathrm{E}} = \frac{\sin\varphi}{\sin(\psi-\varphi)} = \frac{\sin\varphi}{\sin(\theta+\varphi)} \tag{1}$$

Consider a point on the surface at a distance d from the wire projection onto xy plane, and at a distance $r = \sqrt{d^2+h^2}$ from the wire, as shown in Fig. 1(b). It follows from the Ampère's law that the magnitude of magnetic induction of the wire at that point is

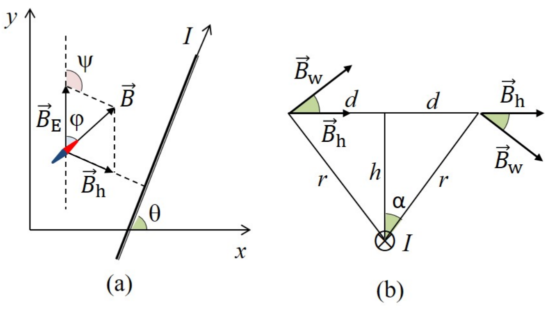
$$B_\mathrm{w} = \frac{\mu_0 I}{2\pi r} \tag{2}$$

and the magnitude of its horizontal projection is

$$B_\mathrm{h} = B_\mathrm{w}\cos\alpha = \frac{\mu_0 I h}{2\pi(d^2+h^2)}. \tag{3}$$

Equations (1) and (3) are sufficient to complete all tasks of the problem.

Figure 1: Notations used in the derivation of the basic equations.



Task (a): Determination of the horizontal position of the wire

As evident from the vector triangle (Fig. 1), the maximum absolute value of the deflection angle at a given current is met at $d = 0$ where B_h is maximal, i.e. vertically above the wire. Therefore, the wire can be tracked by finding two or more points on the surface where $|\varphi|$ reaches a maximum. First, a coarse scan of the border with a step of, say, 10 mm, can be performed in order to locate intervals, where $|\varphi|$ goes through a maximum. In this way we establish that the wire projection crosses the West side ($x = 0$ mm) at $y \in [60\text{ mm}, 90\text{ mm}]$ and the East side ($x = 100$ mm) at $y \in [10\text{ mm}, 30\text{ mm}]$. A finer scan of these intervals with a step of 1 mm allows to determine the approximate coordinates of the two crosspoints as $P_1 = (0.0\pm0.5, 75\pm1)$ mm and $P_2 = (100.0\pm0.5, 17\pm1)$ mm. The uncertainty of the y coordinate is 1 mm since near the maximum $|\varphi|$ changes slowly and takes the same rounded value at three consecutive points. Additional scans along vertical (horizontal) lines of intermediate x (y) values could be done in order to find more points along the wire projection, and to determine the equation of the wire more precisely by means of a least-squares fit. A typical set of values is given in Table 1. The fitted equation of the wire is, respectively

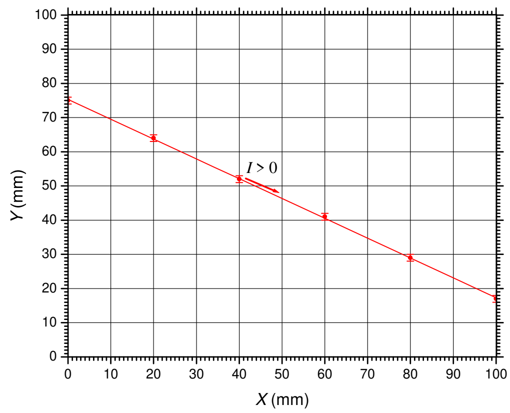
Table 1: Points, where $|\varphi|$ reaches a maximum of 143° at a current $I = +5$ A.

x (mm)	y (mm)
0	75
20	64
40	52
60	41
80	29
100	17

$$y = ax + b = -0.58x + 75.3\text{ mm} \tag{4}$$

with estimated parameter uncertainties of $\delta a \approx 0.01$ and $\delta b \approx 0.4$ mm. The parameter uncertainties scale as $1/\sqrt{N}$, where N is the number of experimental points. A graph of the wire projection in the xy-plane is shown in Fig. 2. Since $\varphi < 0$ at $I > 0$, the positive I direction is from the West to the East border, as shown in the graph.

Figure 2: xy-projection of the wire with indicated positive I direction.



Task (b): Determination of h and B_E

As follows from equations (1) and (3), the deflection angle φ at a distance d from the horizontal projection of the wire satisfies the equation

$$\frac{\sin\varphi}{\sin(\theta+\varphi)} = \frac{\mu_0 I h}{2\pi B_\mathrm{E}(d^2+h^2)}. \tag{5}$$

where the angle θ can be calculated from the slope coefficient a of the wire:

$$\theta = \arctan(a) = -30.1^\circ \pm 0.4^\circ \tag{6}$$

Table 2: Experimental data for the deflection angle φ vs. current I at two different distances d.

(0 mm,75 mm); $d_1 = 0$ mm			(20 mm,75 mm); $d_2 = 10$ mm		
I (A)	φ (deg)	U	I (A)	φ (deg)	U
-5.0	25	-4.85	-5.0	15	-1.00
-4.0	24	-3.89	-4.0	13	-0.77
-3.0	23	-3.21	-3.0	11	-0.59
-2.0	20	-1.97	-2.0	8	-0.37
-1.0	15	-1.00	-1.0	5	-0.21
1.0	-75	1.00	1.0	-7	0.20
2.0	-126	1.99	2.0	-17	0.40
3.0	-137	3.03	3.0	-32	0.60
4.0	-141	4.02	4.0	-52	0.80
5.0	-143	4.94	5.0	-75	1.00
$k_1 = 1.01 \pm 0.01$ A			$k_2 = 5.04 \pm 0.03$ A		

The distance d between a point with coordinates (x, y) and the wire projection can either be measured directly on the graph in Fig. 2, or calculated as:

$$d = |(ax + b - y)\cos\theta| \approx 0.865|ax + b - y| \tag{7}$$

It follows from equations (5)–(7) that the unknown h and B_E could be determined if the deflection angle φ is measured in at least two points situated at different distances from the wire. However, due to the random error, associated with compass positioning and the rounding error of the angle reading, such a minimalist approach is quite inaccurate. Therefore, systematic measurements at several distances d and/or different currents I, are necessary to obtain sufficiently precise estimate for h and B_E. Two generic approaches could be followed, as well as a combination between them.

Method I. Varying the current at fixed distances. By defining a new dimensionless variable $U = \sin\varphi/\sin(\varphi - 30.1°)$, equation (5) is linearized as:

$$I = kU \tag{8}$$

where the slope coefficient is:

$$k = \frac{2\pi B_E(d^2 + h^2)}{\mu_0 h} \tag{9}$$

Therefore, the unknown B_E and h can be estimated after obtaining k for at least two different distances d from the wire. Table 2 summarizes the results of measurements at $d_1 = 0$ mm (vertically above the wire) and at $d_2 = 10$ mm in a point with coordinates $x = 20$ mm and $y = 75$ mm. Figure 3 shows the corresponding U-I graphs, and the estimated values of the slope coefficients are also listed in the table 2.

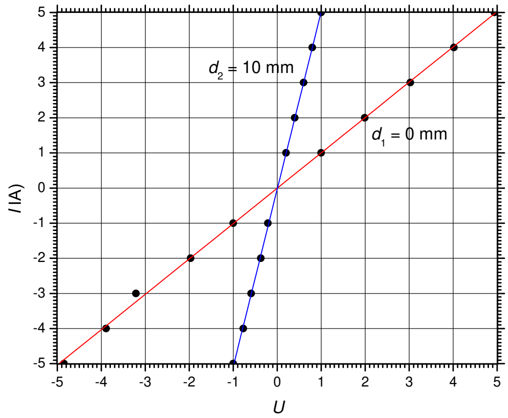
It follows from equation (8) that:

$$\frac{B_E}{h} = \frac{\mu_0(k_2 - k_1)}{2\pi(d_2^2 - d_1^2)} = 8.06 \times 10^{-6}\ \mathrm{T/mm} \tag{10}$$

and

$$B_E h = \frac{\mu_0(d_2^2 k_1 - d_1^2 k_2)}{2\pi(d_2^2 - d_1^2)} = 1.98 \times 10^{-4}\ \mathrm{T \cdot mm} \tag{11}$$

Figure 3: U-I graphs for two different distances d from the wire, and the corresponding linear fits.



Alternatively, one can also use

$$h = \sqrt{\frac{d_2^2 k_1 - d_1^2 k_2}{k_2 - k_1}} = 5.0\ \mathrm{mm}. \tag{12}$$

Finally, we obtain for the horizontal component of the Earth's magnetic induction:

$$B_E = 4.0 \times 10^{-5}\ \mathrm{T} \tag{13}$$

and for the depth of the wire:

$$h = 5.0\ \mathrm{mm} \tag{14}$$

These estimates of h and B_E coincide with accuracy of two significant digits with the values preset in the simulation program.

Method II. Fixed current, varying the distance. Equation (5) can be rewritten in the form:

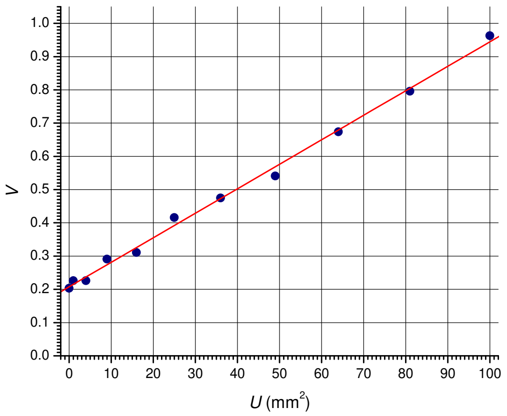
$$\frac{\sin(\theta + \varphi)}{\sin\varphi} = \frac{2\pi B_E}{\mu_0 I h} d^2 + \frac{2\pi B_E h}{\mu_0 I} \tag{15}$$

which can be linearized by setting new auxiliary variables: $U = d^2$ and $V = \sin(\varphi - 30.1°)/\sin\varphi$. A typical data set for this method is given in table 3, while the linearized U-V plot is shown in Fig. 4.

Table 3: Experimental data for the deflection angle φ vs. distance d at a fixed current $I = 5.0$ A.

x (mm)	y (mm)	d (mm)	φ (deg)	U (mm^2)	V
0	75	0	-143	0	0.203
2	75	1	-142	1	0.226
4	75	2	-142	4	0.226
6	75	3	-139	9	0.291
8	75	4	-138	16	0.311
10	75	5	-132	25	0.416
12	75	6	-128	36	0.475
14	75	7	-123	49	0.541
16	75	8	-111	64	0.674
18	75	9	-98	81	0.796
20	75	10	-79	100	0.963

Figure 4: U-V graph obtained at a fixed current $I = 5.0$ A.



From the fitting line we obtain:

$$V = 7.37 \times 10^{-3}\mathrm{mm}^{-2}U + 0.208 \quad (16)$$

which means $2\pi B_\mathrm{E} h/(\mu_0 I) = 0.208$ and $2\pi B_\mathrm{E}/(\mu_0 I h) = 7.37 \times 10^{-3}\mathrm{mm}^{-2}$. Thus, we obtain:

$$B_\mathrm{E} = 3.9 \times 10^{-5}\ \mathrm{T} \quad (17)$$

for the horizontal component of Earth's magnetic induction, and

$$h = 5.3\ \mathrm{mm} \quad (18)$$

for the depth of the wire. These estimates are close to, but less accurate than the values obtained by Method I. The reason is that at small d, there is a large relative error associated with wire positioning, i.e. with variable U. At large d, however, the deflection angle is small, and there is significant relative error, associated with the compass reading, i.e. with V parameter.

Marking scheme

The basic equations could be stated in a separate section of the solution, or spread over different parts of the solution.

Theoretical background		**Points**
T1	States explicitly or shows on a clear graph that the magnetic needle points along the total magnetic induction.	0.1
T2	Derives eq. (1) or equivalent.	0.3
T3	Writes down the Ampere's law (2).	0.2
T4	Derives eq. (3) or equivalent.	0.4
Total on Theory		**1.0**

In tasks A and B points for obtaining final results are given on an additive basis. If a given quantity, say a-parameter of the line, falls into the widest interval, a minimum number of points is given. If the value, however, belongs to the subsequent narrower interval, annotated points are added to the points for the previous interval, and so on, down to the narrowest interval.

Task A: Horizontal position of the wire		**Points**
A1	State or use that the wire is located where $\lvert\varphi\rvert$ is maximal.	0.2
	(state alternative method which allows to find only a);	(0.1)
A2	Find points on the wire at most 2mm away from both cross points with the border.	0.2
A3	Find n points along the wire projection:	
	$n = 3$ or 4;	0.3
	$n \geq 5$.	0.5
A4	Draw wire projection on the graph:	
	plot all measured points or at least 5;	0.2
	if a point drawn incorrectly;	-0.1
	line through the points;	0.2
	axes labels and units;	0.1
	axes tick marks with values;	0.1
	correctly indicated positive I direction.	0.3
A5	Equation of the line:	
	a within [-0.61;-0.55];	0.1
	a within [-0.60;-0.56];	+0.1
	a within [-0.59;-0.57];	+0.3
	b within [73.6;77.0] mm;	0.1
	b within [74.6;76.0] mm;	+0.1
	b within [74.9;75.7] mm .	+0.3
	Correctly estimated uncertainties of a and b.	0.2
Total on Task A		**3.0**

Since there are several approaches to the solution of Task B, the subsequent marking scheme is unified in order to fit all methods of solution. The **data point** is defined as a single measurement of φ at given I, x, and y. The data point **weight** W is defined as a mark related to the way, in which the measured data are presented and treated numerically:

I, x, y, and the corresponding φ are documented in a table with appropriate number of digits.	0.1
The value of the distance d to the wire, and the values of the auxiliary linearizing variables (if required by the solution) are calculated correctly and documented in the table.	0.1
Maximum W	0.2

The total mark for data recording and treatment (B2, see the table below) scales linearly with the number of data points N for up to $N = 8$. All data points after 8-th do not contribute to the total mark on B2. Data points measured in part A only count towards the mark of B2 if it is stated in part B that they can be used for this part as well, or if they are used implicitly.

Task B: Finding B_E and h		**Points**
B1	Makes appropriate choice of auxiliary variables, which linearize eq. (3) OR derives explicit expressions for B_E and h in terms of two measured angles φ at two different distances d (minimalist approach).	0.6
B2	Data recording and treatment:	
	min$(N, 8) \times W$	1.6
B3	Organization of data in table(s):	
	Column titles	0.2
	Units	0.2
B4	Extracting parameters	
	Graphical method:	
	For plotting n points $m =$ min$(n, 8)$	0.1m
	Coverage of at least 75% of the graph window	0.2
	Titles on axes	0.2
	Units on axes	0.2
	Tick marks with annotated values	0.2
	Fitting line(s) is (are) drawn on the graph(s)	0.5
	Fitting line parameters are extracted and explicitly stated	0.5
	Linear regression without graph:	
	using n points $m =$ min$(n, 8)$	0.1m
	correct fit	1.8
	Averaging over n two-point measurements: $m =$ min$(n, 13)$	0.2m
B5	Final values of B_E and h are calculated from the line parameters or calculated from the results of a two-point measurement (minimalist approach):	
	$B_E \in [3.7; 4.3] \times 10^{-5}$ T	0.1
	$B_E \in [3.8; 4.2] \times 10^{-5}$ T	+0.1
	$B_E \in [3.9; 4.1] \times 10^{-5}$ T	+0.2
	$h \in [4.5; 5.5]$ mm	0.1
	$h \in [4.7; 5.3]$ mm	+0.1
	$h \in [4.9; 5.1]$ mm	+0.2
Total on Task B		**6.0**

e

E2: Hot Cylinder

Start the experiment with the heater on full for 300 Watts and the thermostats located evenly across the length of the rod and display the results every 100 seconds. Then plan out the remainder of the experiment while waiting, or do the other experiment. The rod reaches steady state at about 600 seconds. Find the average temperature at the five thermostats by considering the last five measurements; you will use this later.

The most accessible approach is then to study the steady state behavior, the uniform temperature behavior, the low temperature behavior, and the high temperature behavior. Separating the low and high temperature behaviors is useful because blackbody radiation dominates at higher temperatures while convective loss is most significant at near room temperature.

Finding the heat capacity is done by heating the rod at a low enough rate for a short enough time so that heat loss is as small as possible.

One possibility is to give a total of 1500 J of heat, but at various power settings and various times, while keeping the temperature as low as possible.

The average temperature of the rod is computed from the five equally spaced points by applying Simpson's rule,

$$T_{avg} = \frac{T_1 + 4T_2 + 2T_3 + 4T_4 + T_5}{12}$$

Computing instead a direct average yields a $+5\%$ error.

It is found that the average temperature for heating times less than 50 seconds is 55.4 ± 0.5 °C, yielding specific heat capacity of $c = 114 \pm 1$ J/kg K.

Heat the rod full power for 600 seconds, and then allow to cool.

The rod temperature becomes uniform at about 700 seconds. Average the five points to obtain an average rod temperature.

Linear cooling predicts a straight line graph for $\ln(T - T_0)$ versus t

The convective heat loss rate is then given by $A\alpha(T - T_0)$, where A is the surface area of the rod. Do not forget the end caps!

The radiative heat loss rate is $\beta\sigma(T^4 - T_0^4)$ where $\sigma = 5.67 \times 10^{-8}$ W/(m^2 K^4). The radiative heat loss rate is then given by $A\beta\sigma(T^4 - T_0^4)$.

Note that at temperatures close to T_0 the radiative expression can be written as

$$A\beta\sigma(T^4 - T_0^4) \approx A\beta\sigma 4(T - T_0)T_0^3$$

This means that the linear heat loss rate at temperatures close to T_0 is

$$A\left(\alpha + \beta\sigma 4T_0^3\right)(T - T_0)$$

For the uniform, low temperature cooling rod,

$$mc\frac{dT}{dt} = -A\left(\alpha + \beta\sigma 4T_0^3\right)(T - T_0)$$

The solution is of the form

$$T - T_0 = Ce^{-Bt}$$

where

$$B = A\frac{\alpha + \beta\sigma 4T_0^3}{mc}$$

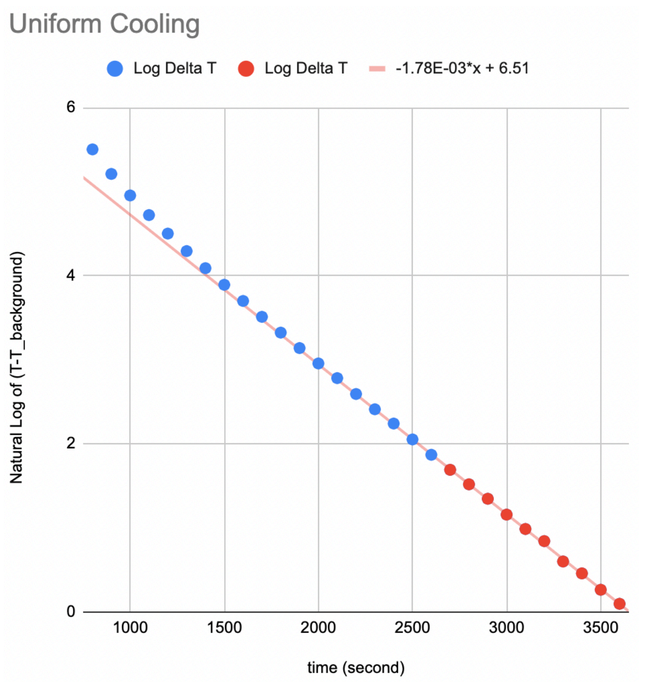
On a log plot of $\ln(T - T_0)$ as a function of time t, the plot should be linear, with a slope given by

$$-A\frac{\alpha + \beta\sigma 4T_0^3}{mc}$$

It is also possible to plot dT/dt as a function of $T - T_0$, and the plot will be linear, with a slope also given by

$$-A\frac{\alpha + \beta\sigma 4T_0^3}{mc}$$

The slope in either case is found to be -1.78×10^{-3} /s.



Note that only the last points (in red) were used to determine the linear cooling line. It is clearly a good fit from $t = 2000$ s on, which corresponds to rod temperatures of $T < 45$ C.

To find the blackbody behavior we want to heat the rod as much as possible such that the blackbody heating becomes the dominant form of heat loss. Since the hot rod is in steady state, the heat radiated must be equal to 300 W. Use the results from the beginning.

The average temperature of the rod is computed from the five equally spaced points by

$$T_{avg} = \frac{T_1 + 4T_2 + 2T_3 + 4T_4 + T_5}{12} = 662\,°\text{C}.$$

Computing a direct average yields a $+1.5\%$ error.

The average of T^4 is found from

$$T_{\text{avg}}^4 = \frac{{T_1}^4 + 4{T_2}^4 + 2{T_3}^4 + 4{T_4}^4 + {T_5}^4}{12} = 7.95 \times 10^{11}\ \text{K}^4$$

Computing a direct average yields a $+6.3\%$ error.

The rate of linear temperature heat loss is found from above to be

$$(-1.78 \times 10^{-3}\ /\text{s})mc\Delta T = 59\ \text{W}$$

The blackbody remainder term is then

$$300 - 59 = 241\ \text{W},$$

and necessarily equals

$$A\beta\sigma(T^4 - T_0^4) - A\beta\sigma 4T_0^3(T - T_0),$$

where the second term reflects the fact that we had considered part of the blackbody behavior as being linear.

Solving, $\beta = 0.304 \pm 0.004$.

Failing to subtract the second term would yield $\beta = 0.28$.

We are now in a position to find α, from

$$-A\frac{\alpha + \beta\sigma 4T_0^3}{mc} = -1.78 \times 10^{-3}\ /\text{s}$$

which yields $\alpha = 2.93$

Alternatively, for the uniform, high temperature cooling rod,

$$mc\frac{dT}{dt} \approx -A\beta\sigma\left(T^4 - T_0^4\right)$$

as the radiative cooling effect will dominate.

On a plot of dT/dt as a function of $T^4 - T_0^4$, the plot should be linear, with a slope given by

$$-\frac{A\beta\sigma}{mc}$$

The slope is found to be -7.8×10^{-12} K^3/s

This means $\beta/c = 3.25 \times 10^{-3}$ kg K/J; this gives $\beta = 0.36$, which is too high; ignoring the linear loss effects was significant; as was previously seen, almost 20% of the heat loss is from convection in this temperature range.

We can use the high temperature behavior to find the heat flux through the center of the rod. The average of T and T^4 on the non-heated half of the rod is 599 C and 5.8×10^4 K^4, yielding a heat loss at 112 W. That heat necessarily came from the other side of the rod.

The temperature gradient is -898 K/m, so $k = 397$ W/mK. Don't forget that the formula provided gave the rate of heat flux, which means that we needed to consider the cross sectional area of the wire.

Marking scheme

Finding c, 2.5 pt total

	Task	Pts
2.1	Idea of heating the rod by a fixed Q.	0.6
2.2	Obtaining an equation relating the inserted heat with the temperature change and c.	0.4
2.2	Heating the rod for a short duration for mitigating the effects of heat loss:	
	heating for less than 60 seconds.	0.2
2.3	Averaging the temperature of the rod:	
	averaging over one to three points;	0.1
	Use Simpson rule (or equivalent)	0.3
	averaging over four or more points;	0.2
2.4	Checking more than one time value	0.2
2.5	Numerical value of c:	
	c within [103;123] J/(K kg);	0.3
	c within [108;118] J/(K kg).	0.2

Finding the linear heat loss, 2.0 pt total

	Task	Pts
3.1	Idea of looking at how the rod cools down at the low temperature limit (with no heating).	0.2
3.2	Obtaining an equation for T as a function of t in terms of α, β, and c:	
	linearize radiative loss around T_0;	0.3
	obtaining a differential equation for $T(t)$;	0.1
	solving the differential equation to get $T(t)$.	0.2
3.3	Finding the average temperature at t:	
	averaging over one to three points;	0.1
	Use Simpson rule (or equivalent)	0.3
	averaging over four or more points;	0.1
3.4	Graphically finding the slope (which is a function of α, β, and c):	
	Plot 2 to 4 points in range $T < 50$C;	0.1
	Plotting 5 or more points in range $T < 50$C;	0.1
	axes labels and units;	0.1
	axes tickmarks with values.	0.1
3.5	Numerical value of the slope:	
	slope within [-1.58×10^{-3},-1.98×10^{-3}];	0.2
	slope within [-1.68×10^{-3},-1.88×10^{-3}];	0.1

Finding β, 2.5 pt total

	Task	Pts
4.1	Idea of looking at the steady state at the high temperature limit.	0.2
4.2	Writing down the heat balance:	
	accounting for the area of the end caps;	0.1
	accounting for the linear contribution to the heat loss by removing the α dependence from the previously found slope;	0.2
	final expression for β in terms of mean value of T and T^4 of the steady state. Making a mistake in the previous parts shouldn't be penalised here.	0.2
4.3	Measurements:	
	Heating power sufficiently big such that the steady state temperature is bigger than 500 °C;	0.1
	Waiting for at least 300s to reach the steady state	0.1
	Waiting for at least 600s to reach the steady state	0.1
4.4	Finding the average temperature:	
	averaging over one to three points;	0.1
	Use Simpson rule (or equivalent)	0.3
	averaging over four or more points;	0.1
4.5	Finding the average T^4 (for calculating average radiative loss):	
	averaging over one to three points;	0.1
	Use Simpson rule (or equivalent)	0.3
	averaging over four or more points;	0.1
4.6	Numerical value of β:	
	β within [0.25;0.35];	0.3
	β within [0.28;0.32].	0.2

Finding α, 0.5 pt total

	Task	Pts
5.1	Obtaining an expression for α in terms of the slope γ.	0.1
5.2	Numerical value of α:	
	α within [2.33;3.23]W/(m^2 K);	0.2
	α within [2.53;3.03]W/(m^2 K).	0.2

Finding k, 2.5 pt total

	Task	Pts
6.1	Idea of looking at the flux from part of the rod to the other	0.4
6.2	Theory:	
	Expressing heat flux in terms of k and the temperature gradient;	0.2
	Expressing heat flux in terms of the average T, T^4, and the heating power of one of the halves of the rod;	0.4
	accounting for the area of the end caps.	0.1
6.3	Finding the average temperature of one of the halves:	
	averaging over one to three points;	0.1
	averaging over four or more points. No marks if points not equally spaced and average doesn't account for the unevenness;	0.1
6.4	Finding the average T^4 (for calculating average radiative loss):	
	averaging over one to three points;	0.1
	averaging over four or more points;	0.1
6.5	Finding the temperature gradient:	
	Using at least two points for the gradient calculation;	0.1
	Using $(f(x+h) - f(x-h))/2h$ for numerical derivative;	0.2
	Having the range of points used for gradient calculations not farther apart than 5 cm;	0.1
	Having the range of points used for gradient calculations not closer than 1 cm;	0.1
6.5	Numerical value of k:	
	k within [328;488]W/(m K);	0.3
	k within [378;438]W/(m K).	0.2

Some grading notes:

- Failure to record and report the location of the sensors will result in a penalty of **-1.0 pt** *for each occurrence!*. It is acceptable to clearly state the location of the sensors in one part of the report, and then mentioning that they are not moved during the experiment.
- When computing spatial averages, if the spacing between thermometers is not uniform, the averaging techniques must use appropriate weighting, or there is a penalty of **-0.1 pt** *for each occurrence!*
- When computing spatial averages, if the rod is not mostly uniform in temperature, a Simpson's Rule technique or equivalent must be used to obtain the 0.3 pts. If instead all of the temperatures are within two error limits, then Simpson is not required to obtain the 0.3 pt.
- Any numerical derivatives must use the symmetric form

$$f'(x) \approx (f(x+h) - f(x-h))/2h$$

or some equivalent, or better, method.