General guidelines for marking

- Granularity for marks is 0.1 p.
- A simple numerical error resulting from a typo is punished by 0.2 p unless the grading scheme explicitly says otherwise.
- Errors which cause dimensionally wrong results are punished by at least 50 % of the marks unless the grading scheme explicitly says otherwise.
- Propagating errors are not punished repeatedly unless they either lead to considerable simplifications or wrong results whose validity can easily be checked later.

T1: Floating cylinder

Solution I: energetic approach

Denote the density of the liquid by ϱ, so the density of the cylinder is $\gamma\varrho$. In equilibrium (i.e. when the net force acting on the cylinder is zero) the immersed part of the cylinder has height γh.

Consider the system in a moment when the cylinder is displaced by distance x_1 downward and moves down with velocity v_1. As a result of the motion of cylinder the liquid level rises by some height x_2, and the liquid flows in the gap between the cylinder and beaker with some velocity v_2 upwards (see *Fig. 1*).

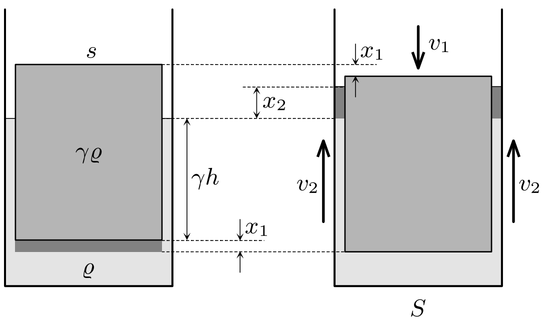


Fig. 1

The relation between the aforementioned displacements and velocities are given by the continuity law:

$$x_1 s = x_2(S-s), \qquad v_1 s = v_2(S-s)\,.$$

In the following we express the potential and kinetic energy of the system. Compared to the equilibrium position the cylinder of mass $\gamma\varrho sh$ sunk by x_1, while the potential energy change caused by the redistribution of liquid can be imagined as the center of mass of liquid with mass $\varrho s x_1$ rises by distance $\gamma h + x_1/2 + x_2/2$. Taken the potential energy in the equilibrium state to be zero, the potential energy in the state indicated in the right figure can be written as

$$E_{\text{pot}} = -\gamma\varrho shgx_1 + \varrho s x_1 g\left(\gamma h + \frac{x_1+x_2}{2}\right)\,.$$

After opening the bracket the first two terms cancel each other:

$$E_{\text{pot}} = \frac{1}{2}\varrho sgx_1(x_1+x_2)\,.$$

After expressing x_2 from continuity law and some simplification we get a quadratic expression for the potential energy:

$$E_{\text{pot}} = \frac{1}{2}\varrho sgx_1\left(x_1 + \frac{s}{S-s}x_1\right) = \frac{1}{2}\varrho\frac{sS}{S-s}gx_1^2\,.$$

Now let us calculate the kinetic energy of the system. The contribution from the cylinder is straightforward, $\gamma\varrho shv_1^2/2$, but the motion of the liquid is more complicated.

Note. We may notice that since $s/(S-s) = 50$, the speed v_2 of the liquid in the narrow gap is 50 times larger than the typical speed of the liquid below the cylinder (which can be estimated to be in the range of v_1). And while the mass of the liquid below the cylinder is much larger than the mass of liquid inside the gap (the ratio is ca. 25 if the „few centimeters" in the problem text is taken to be 3.5 cm), the kinetic energy is proportional to the square of the velocity, so the kinetic energy of the liquid inside the gap is roughly 100 times larger than the kinetic energy of the liquid below the cylinder.

Since the kinetic energy of the liquid below the cylinder is negligible, we can write the total kinetic energy of the system as:

$$E_{\text{kin}} = \underbrace{\frac{1}{2}\gamma\varrho shv_1^2}_{\text{cylinder}} + \underbrace{\frac{1}{2}\varrho(S-s)\left(\gamma h + x_1 + x_2\right)v_2^2}_{\text{liquid}}\,.$$

Here $x_1, x_2 \ll \gamma h$, so we shall keep only the term containing γh in the second bracket:

$$E_{\text{kin}} = \frac{1}{2}\gamma\varrho shv_1^2 + \frac{1}{2}\varrho(S-s)\gamma hv_2^2$$

Expressing v_2 from continuity law gives the following:

$$E_{\text{kin}} = \frac{1}{2}\gamma\varrho shv_1^2 + \frac{1}{2}\varrho\gamma h\frac{s^2}{S-s}v_1^2 = \frac{1}{2}\varrho\gamma h\frac{sS}{S-s}v_1^2\,.$$

The potential and kinetic energies can be written in the form

$$E_{\text{pot}} = \frac{1}{2}k_{\text{eff}}\,x_1^2\,, \qquad E_{\text{kin}} = \frac{1}{2}m_{\text{eff}}\,v_1^2\,,$$

where the effective spring constant and effective mass are given by

$$k_{\text{eff}} = \varrho\frac{sS}{S-s}g\,, \qquad m_{\text{eff}} = \varrho\gamma h\frac{sS}{S-s}\,.$$

So the oscillation is indeed harmonic, thus the angular frequency and the period are:

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g}{\gamma h}}, \qquad T = 2\pi\sqrt{\frac{\gamma h}{g}} = 0.53\ \text{s}\,.$$

Note. The static restoring force, acting on the cylinder is due to the change (relative to the equilibrium position) of the hydrostatic pressure at its lower base:

$$F = -s\rho g(x_1+x_2) = -\frac{sS}{S-s}\rho gx_1.$$

This immediately gives effective stiffness of the system $k_{\text{eff}} = \frac{sS}{S-s}\rho g$.

Alternatively, one may wish to integrate $\int F\,\mathrm{d}x_1$ to get the potential energy

$$E_{\text{pot}} = \frac{sS}{S-s}\frac{\rho g}{2}x_1^2.$$

Solution II: dynamical approach

When the cylinder is displaced from its equilibrium position downwards by distance x_1, the net restoring force (pointing up) can be calculated as the sum of the weight of the cylinder and the force from the difference of pressures at the top (p_0) and bottom (p) of the cylinder. As a result of the net force, the cylinder accelerates upwards with a_1, and at the same time, the liquid located in the gap between the cylinder and the wall of the beaker accelerates down with a_2. The relation between the magnitudes of a_1 and a_2 is given by the continuity law:

$$sa_1 = (S - s)a_2 \,.$$

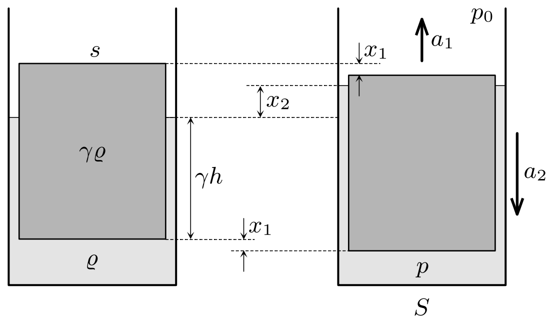


Fig. 2

If the liquid in the gap was not accelerating, the pressure difference $p - p_0$ would be equal to the hydrostatic pressure of the liquid column in the gap. Due to the acceleration of the liquid, $p - p_0$ can be expressed from Newton's 2nd law applied for the liquid column of unit area located in the gap:

$$p_0 - p + \varrho g(\gamma h + x_1 + x_2) = \varrho(\gamma h + x_1 + x_2)a_2 \,,$$

where we used the notations of *Solution I*, and the downward direction was taken as positive.

Newton's 2nd law for the cylinder reads as

$$(p - p_0)s - \gamma\varrho shg = \gamma\varrho sha_1 \,.$$

After expressing $p - p_0$ from the previous equation, and then substituting it here we get:

$$\varrho g(\gamma h+x_1+x_2)s-\varrho(\gamma h+x_1+x_2)a_2 s-\gamma\varrho shg = \gamma\varrho sha_1 \,.$$

Since the amplitude of the liquid level is small, the terms containing a_2x_1 and a_2x_2 can be neglected. After rearranging we get:

$$\varrho gs(x_1 + x_2) = \gamma\varrho sh(a_1 + a_2) \,.$$

Using the relations between the displacements and accelerations we finally get:

$$a_1 = \frac{g}{\gamma h}x_1 \,.$$

Taking into account the opposite directions of x_1 and a_1, this is the dynamical condition of a simple harmonic motion with angular frequency and period

$$\omega = \sqrt{\frac{g}{\gamma h}}\,, \qquad T = 2\pi\sqrt{\frac{\gamma h}{g}} = 0.53\,\text{s}\,.$$

Note. In this solution we assumed that the pressure p is constant throughout the bottom surface of the cylinder. This assumption is equivalent with saying that the horizontal acceleration of the liquid below the cylinder at every point is much smaller than a_2, which is reasonable.

Marking scheme

All solutions should be graded according to only one marking scheme (either energetical or dynamical). If the student used both ideas, that marking scheme should be used which results in a higher score.

Solution I: energetic solution		**pts**
i	Height of submerged part of cylinder in equilibrium is γh.	0.5
ii	Realizing that the kinetic energy of water is important	1.0
iii	Realizing that the kinetic energy of liquid below the cylinder is negligible	1.5
iv	Expressing the kinetic energy of liquid inside the gap as a function of velocity of cylinder.	2.5
v	Potential energy change of liquid as a function of the small displacement of cylinder	1.0
vi	Potential energy change (0.5 p) and kinetic energy change of cylinder (0.5 p)	1.0
vii	Continuity law either for displacements or velocities (only 0.5 p if the factor is $S/(S-s)$)	1.0
viii	Expressing ω from the formulas for E_{pot} and E_{kin} ($\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$ or equivalent).	1.0
ix	$T = 2\pi/\omega$	0.3
x	Correct substitution of values, final result	0.2
	Total number of points	**10.0**

Solution II: dynamical solution		**pts**
I	Height of submerged part of cylinder in equilibrium is γh	0.5
II	Realizing that the pressure difference between top and bottom of the cylinder is *not* $\varrho g \times$ height diff.	1.0
III	Neglecting the motion of water below the cylinder but not on the sides	1.5
IV	Newton's 2nd law for liquid in the gap with nonzero acceleration. (0 p for $p - p_0 = \varrho g \times$ height diff.)	2.5
V	Newton's 2nd law for cylinder (still full mark if II was not realized but $p - p_0$ was used properly)	1.0
VI	Using the change in water level in Newton's 2nd law	1.0
VII	Continuity law either for displacements or accelerations (only 0.5 p if the factor is $S/(S-s)$)	1.0
VIII	Concluding a linear relation between acceleration and displacement of cylinder	0.5
IX	Expressing ω from the dynamical equations (expressing $\omega = \sqrt{k_{\text{eff}}/m_{\text{eff}}}$ correctly or equivalent).	0.5
X	$T = 2\pi/\omega$	0.3
XI	Correct substitution of values, final result	0.2
	Total number of points	**10.0**

T2: Thermal oscillations

Part (a): Critical voltages

The power heating the resistor is $P_{\text{el}} = V^2/R_j$. The thermal equilibrium is reached when $P_{\text{el}} = P = \alpha(T_{\text{eq}} - T_0)$. To avoid oscillations, the equilibrium temperature T_{eq} must satisfy $T_{\text{eq}} < T_c$ if $R = R_1$ and $T_{\text{eq}} > T_c$ if $R = R_2$. Solving for V, we have

$$V = \sqrt{R_j \alpha (T_{\text{eq}} - T_0)}. \tag{1}$$

The critical values therefore are

$$V_1 = \sqrt{R_1 \alpha (T_c - T_0)} \quad \text{and} \quad V_2 = \sqrt{R_2 \alpha (T_c - T_0)}. \tag{2}$$

Part (b): Temperature behaviour

In the oscillating regime, we have a time-dependent current $I(t)$. The power dissipated over the resistor is $P_{\text{el}}(t) = R(t)I(t)^2$. By assumption (ii), we may assume that the thermal equilibrium is reached very fast, i.e. $P_{\text{el}}(t) = P(t)$. The temperature $T(t)$ is therefore determined by the current via

$$T(t) = T_0 + \frac{R(t)I(t)^2}{\alpha}. \tag{3}$$

If the resistance has value R_1, the current will increase, trying to reach $J_1 = V/R_1$. The difference $I(t) - V/R_1$ will decay exponentially, with characteristic time L/R_1. The phase transition occurs once the critical current

$$I_1 = \sqrt{\frac{\alpha(T_c - T_0)}{R_1}}$$

is reached. After the phase transition, the current will decrease, approaching the new equilibrium value $J_2 = V/R_2$. Again, $I(t) - V/R_2$ will decay exponentially with characteristic time L/R_2, until the critical current

$$I_2 = \sqrt{\frac{\alpha(T_c - T_0)}{R_2}}$$

is reached. This behaviour is shown in Fig. 1.

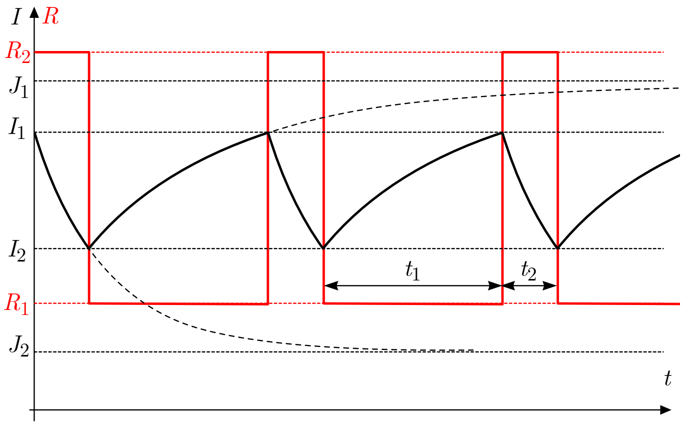


Fig. 1

Together with (3), we see that the temperature behaves like in Figure 2.

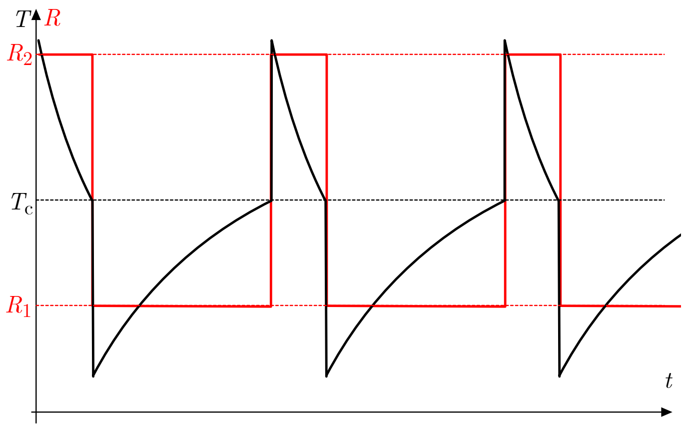


Fig. 2

The maximum and minimum temperatures will be attained just after the phase transitions occur. We obtain that

$$\frac{T_{\text{max}} - T_0}{T_{\text{min}} - T_0} = \frac{R_2 I_1^2}{R_1 I_2^2} = \frac{R_2^2}{R_1^2}. \tag{4}$$

Part (c): Period of oscillations

If the phase transition occurs at $t = 0$, with the resistance changing from $R_{j'}$ to R_j, the current is given by

$$I(t) = \frac{V}{R_j} + \left(I_{j'} - \frac{V}{R_j}\right) \mathrm{e}^{-R_j t/L} \tag{5}$$

until the next phase transition occurs when $I(t_j) = I_j$. Hence, the period is

$$t_1 + t_2 = \frac{L}{R_1} \ln\left(\frac{I_2 - V/R_1}{I_1 - V/R_1}\right) + \frac{L}{R_2} \ln\left(\frac{I_1 - V/R_2}{I_2 - V/R_2}\right) \tag{6}$$

Inserting the relations $R_2 = \eta R_1$ and $V = \sqrt{V_1 V_2} = \eta^{1/4}\sqrt{R_1 \alpha (T_c - T_0)}$, we obtain the period

$$\frac{L}{R_1} \ln\left(\frac{7}{4}\right) + \frac{L}{R_2} \ln(7) = \frac{L}{R_1}\left(\ln\left(\frac{7}{4}\right) + \frac{1}{16}\ln(7)\right) \approx 0.68\frac{L}{R_1}. \tag{7}$$

Marking scheme

Task (a): Critical voltages		**pts**
a1	Formula for the power dissipation $P_{\text{el}} = V^2/R_j$.	0.5
a2	Relating the power dissipation to the temperature of the resistor in oscillations-free stationary regime, $P_{\text{el}} = P = \alpha(T_{\text{eq}} - T_0)$	0.5
a3	Expressing the voltage in terms of the temperature if the thermal equilibrium were to be reached, $V = \sqrt{R_j \alpha(T_{\text{eq}} - T_0)}$. Subtract 0.1 pts if V is not expressed explicitly.	0.5
a4	Realising that oscillations will not happen if $V > \sqrt{R_2 \alpha(T_{\text{eq}} - T_0)}$ or $V < \sqrt{R_1 \alpha(T_{\text{eq}} - T_0)}$. No marks if only one inequality is obtained (but no subtractions because of that in a3 - in most cases those who got correct expression for one of the voltages but has a wrong or missing expression for the other gets full marks for a1-a3, and 0 pts for a4).	0.5
	Total number of points for Task (a)	**2.0**

Task (b): Temperature behavior		**pts**
b1	Realising that the $I - t$ curve is made of segments of exponents, joined without discontinuities. Partial credit of 0.5 pts if it is made of curved segments for which it is not clear that these are exponents, or if these are growing exponents, but which are connected continuously with a discontinuous derivative $\frac{dI}{dt}$. No points if $I(t)$ is discontinuous, or if only one segment of an exponent is shown. Full marks can be given if there is no $I-t$ graph, but the $T-t$ graph is made of the segments of vanishing exponents, connected with temperature jumps in a correct direction, and a partial credit of 0.5 pts if the segments of the $T-t$ are either growing exponents or curves of unclear shape, still connected so that it would correspond to a continuous $I(t)$-curve with a discontinuous derivative. Partial credit of 0.5 pts is given if there is no $I-t$-curve shown, but $V-t$ curve is shown to be made of decaying exponential segments, connected with jumps	1.0
b2	Realising that (i) one of these exponents is in a form $a_1 - b_1 e^{-t/\tau_1}$ and (ii) the other one — in a form $a_2 + b_2 e^{-t/\tau_2}$ where (iii) the $a_1 > a_2$ and (iv) $\tau_1 > \tau_2$. It is not necessary to write down these inequalities mathematically — it is enough it these are clear from a sketch. Inequality $\tau_1 > \tau_2$ does not need to be written if expressions for τ_1 and τ_2 are given. Full marks can be given if $I-t$ graph is missing, but $T-t$ graph is correct and has *all the features* as described in b6. Full marks can be also given if the correct exponential forms are documented not here, but in part c.	0.3+ 0.3+ 0.3+ 0.1
b3	Realising that this exponential behaviour breaks down once the critical temperature is reached. This does not need to be written specifically if the *jumps* in $T-t$ graph happen at $T = T_c$. No marks are given if there is no clear discontinuity of T at T_c and/or if there are discontinuities of $T(t)$ or $\frac{dT}{dt}$ at some other values of T.	1.0
b4	Relating the critical temperature to the corresponding critical current I_j	0.5
b5	Realising that the temperature curve $T(t)$ is related to $I(t)$-curve, $T(t) = T_0 + \frac{R(t)I(t)^2}{\alpha}$	0.5
b6	Drawing a correct final sketch which has the following features: exponential segments showing an exponential relaxation of $T(t)$ in a right direction both when $R = R_1$ and when $R = R_2$; jumps in a right direction each time when T reaches T_c (subtract 0.2 for each missing label on the axes and also if the temperature jumps do not occur at the same value of T). No points are given if *any* of the listed features is missing.	1.0
b7	Using the feature from the graph that the maximal and minimal temperatures are taken immediately after a phase transition when $I = I_1$ and $I = I_2$	0.5
b8	Correct answer for the ratio of the maximal and minimal temperatures. Only 0.3 pts if the answer is not simplified.	0.5
	Total number of points for Task (b)	**6.0**

Task (c): Period of oscillations		**pts**
c1	Expressing the duration of each of the exponential segments as $t_j = \frac{L}{R_j} \ln \frac{\Delta I_{j,i}}{\Delta I_{j,f}}$ where $\Delta I_{j,i}$ and $\Delta I_{j,f}$ denote the corresponding initial and final departures of the current from the equilibrium value (full marks to be given if the final answer is correct). Subtract 0.2 for each incorrect $\Delta I_{j,i}$ and $\Delta I_{j,f}$, $i = 1, 2$ (this means that if none of them is correct, only 0.2 pts are given for c1). 60% of points if t_j is related to $\Delta I_{j,i}$ and $\Delta I_{j,f}$ correctly, but not expressed explicitly.	0.5+ 0.5
c2	Correct first and second terms in the final answer (40% of it if the answer is not simplified)	0.5+ 0.5
	Total number of points for Task (c)	**2.0**

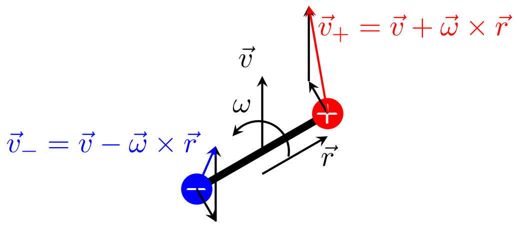
T3: Dipole in a magnetic field

Part (a): Uniform linear motion

Lorentz forces acting on the charges:

$$\vec{F}_+ = q\vec{v}_+ \times \vec{B} = q(\vec{v} + \vec{\omega} \times \vec{r}) \times \vec{B},$$
$$\vec{F}_- = (-q)\vec{v}_- \times \vec{B} = (-q)(\vec{v} - \vec{\omega} \times \vec{r}) \times \vec{B},$$

where $\vec{r}$ is a vector from the center of mass to the position of the positive charge.



According to Newton's first law, the center-of-mass C of the dipole will move with constant velocity provided that the net force:

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{v}_+ - \vec{v}_-) \times \vec{B}, \tag{8}$$

acting on the dipole, is zero. Since $\vec{v}_+$,$\vec{v}_-$ and $\vec{B}$ are perpendicular, we require $\vec{v}_+ = \vec{v}_-$. It means that dipole does not rotate: $\omega = \omega_0 = 0$.

The pure translation, however, is possible if the pair of forces $\vec{F}_+$, $\vec{F}_-$, has zero torque about C:

$$\vec{\tau} = \vec{r} \times \vec{F}_+ - \vec{r} \times \vec{F}_- = 2q\vec{r} \times (\vec{v} \times \vec{B}) =$$
$$2q\left(\vec{v}(\vec{r} \cdot \vec{B}) - \vec{B}(\vec{r} \cdot \vec{v})\right) = -2q\vec{B}(\vec{r} \cdot \vec{v}). \tag{9}$$

We conclude that scalar product is zero only when $\vec{v} \perp \vec{r}$, i.e. the initial velocity should be parallel to Y direction.

In summary, the dipole will move uniformly along Y if, and only if, $\vec{v}_0 \| Y$ and $\omega_0 = 0$.

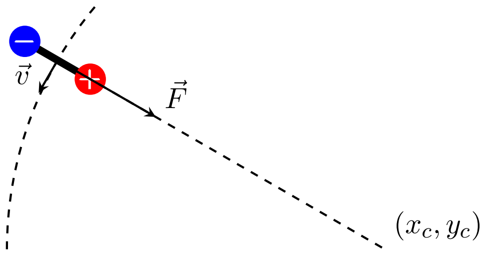
Part (b): Circular motion

The net force can be calculated as:

$$\vec{F} = \vec{F}_+ + \vec{F}_- = 2q(\vec{\omega} \times \vec{r}) \times \vec{B} =$$
$$-2q\left(\vec{\omega}(\vec{B} \cdot \vec{r}) - \vec{r}(\vec{B} \cdot \vec{\omega})\right) = 2qB\omega\vec{r} = B\omega\vec{p}, \tag{10}$$

where $\vec{p}$ is a dipole moment ($|\vec{p}| = qd = 2qr$ and the direction aligns with $\vec{r}$).

When C orbits a circle, $\vec{F}$ acts as a centripetal force, i.e. it points to the center of the circle. Since $\vec{F} \| \vec{p}$, the dipole is always in line with the center of the orbit. Therefore, the orbital angular velocity of C is equal to the angular velocity of rotation of the dipole about C.



The magnitude of the orbital velocity is:

$$v_0 = |\omega_0| R_c$$

From Newton's second law, and accounting that the total mass of the dipole is $2m$:

$$\frac{2mv_0^2}{R_c} = \frac{pBv_0}{R_c},$$

i.e. the magnitude of velocity is:

$$v_0 = \frac{pB}{2m} = \frac{qBd}{2m}$$

and the radius of the orbit is:

$$R_c = \frac{v_0}{|\omega_0|} = \frac{qBd}{2m|\omega_0|}$$

The coordinates of the center of the circle are:

$$(x_c, y_c) = (\pm R_c, 0)$$

where the "$+$" sign corresponds to $\omega_0 > 0$, i.e. counter-clockwise rotation, and the "$-$" sign —to clockwise rotation. In either case, the initial velocity should point to the negative Y direction:

$$\vec{v}_0 = -\frac{qdB}{2m}\hat{\jmath}.$$

Part (c): Reversal of the dipole

In (10) we have shown that the net force:

$$\vec{F} = 2q(\vec{\omega} \times \vec{r}) \times \vec{B} = (\vec{\omega} \times \vec{p}) \times \vec{B}.$$

Since the dipole moment $\vec{p}$ rotates with angular velocity $\vec{\omega}$, its time derivative:

$$\frac{d\vec{p}}{dt} = \vec{\omega} \times \vec{p}.$$

From Newton's second law:

$$2m\frac{d\vec{v}}{dt} = \vec{F} = \frac{d\vec{p}}{dt} \times \vec{B}.$$

By integrating the equation, we arrive at an additional conservation law in the system (conservation of the so called "generalized momentum"):

$$2m\vec{v} - \vec{p} \times \vec{B} = \text{const}$$

Thus, if $\vec{p}$ has reversed its direction from $\vec{p}_0$ to $\vec{p}_1 = -\vec{p}_0$, then the velocity:

$$\vec{v}_1 = \vec{v}_0 + \frac{(\vec{p}_1 - \vec{p}_0) \times \vec{B}}{2m} = -\frac{\vec{p}_0 \times \vec{B}}{m}. \tag{11}$$

Since the magnetic field does not perform work on moving electric charges, the kinetic energy of the dipole is conserved:

$$\frac{I}{2}\omega_0^2 = \frac{I}{2}\omega_1^2 + \frac{2m}{2}v_1^2,$$

Here, $I = 2 \times m(d/2)^2 = md^2/2$ is the moment of inertia of the dipole with respect to its center-of-mass. Since v_1 doesn't depend on angular velocities, ω_0 is minimal when $\omega_1 = 0$. Finally,

$$\omega_{\min} = v_1\sqrt{\frac{2m}{I}} = \frac{p_0 B}{m}\sqrt{\frac{4}{d^2}} = \frac{2qB}{m}$$

Alternatively, we can introduce θ to be the angle between the dipole moment and the axis X ($\theta_0 = 0$) and rewrite the equations of translational motion in coordinates using $\omega = \dot\theta$:

$$\dot v_x = \dot\theta\frac{qBd}{2m}\cos\theta, \qquad \dot v_y = \dot\theta\frac{qBd}{2m}\sin\theta.$$

By integrating these equations, given zero initial velocity, we find how velocity depends on θ:

$$v_x = \frac{qBd}{2m}\sin\theta, \qquad v_y = \frac{qBd}{2m}(1-\cos\theta).$$

Using the expression (9) for the torque, we can write the equation of rotational motion as:

$$I\ddot\theta = \tau = -2qB(r_x v_x + r_y v_y) = -\frac{q^2B^2d^2}{2m}\sin\theta,$$

$$\ddot\theta + \frac{q^2B^2}{m^2}\sin\theta = 0, \qquad (12)$$

This is the equation of a mathematical pendulum of length L in gravitational field $g = L(qB/m)^2$. And the equivalent question becomes what is the minimal push $\dot\theta_0$ required in the bottom position for the pendulum to reach the top position. Kinetic energy of the pendulum $K = \frac{1}{2}mL^2\dot\theta_0^2$ will be transfered to the potential energy $U = 2mgL$, from which we find:

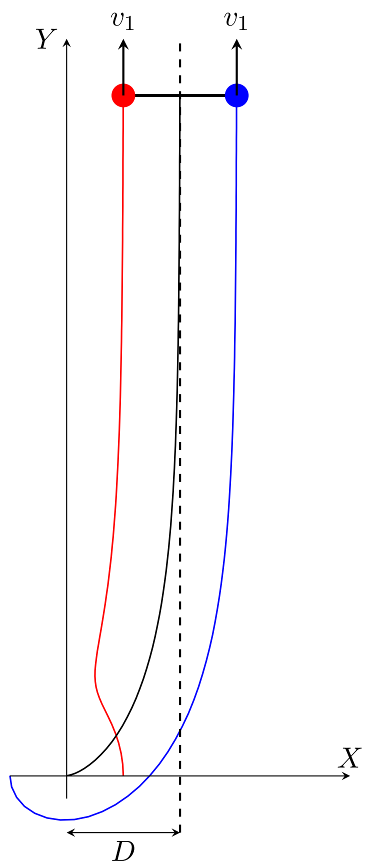
$$\omega_{\min} = \dot\theta_0 = \sqrt{4\frac{g}{L}} = 2\frac{qB}{m}.$$

Note. Due to symmetry, both clockwise and counter-clockwise initial rotation with absolute value of $|\omega_0|$ will work.

Part (d): Trajectory asymptote

If dipole's trajectory has an asymptote, then its movement along the asymptote is uniform. Indeed, if there is a linear motion with acceleration, the dipole $\vec p$ should be always aligned with the direction of motion, thus, not rotating. and as we found in part (a), the absence of rotation can only be maintained if $\vec v =$ const and $\vec v \perp \vec p$.

The uniform linear motion requires $\omega = 0$, and this happens in the limit when the orientation is reversed $\vec p_1 = -\vec p_0$. According to (11), in the limit, the dipole is travelling with the speed $\vec v_1 = p_0 B\hat\jmath/m$. Thus the asymptote is parallel to Y axis: $x = D$ (for counter-clockwise initial rotation).



If $\vec R_+$ and $\vec R_-$ are absolute positions of the charges, we can write equation for the angular momentum around the origin L_O:

$$\frac{d\vec L_O}{dt} = \vec R_+ \times (q\dot{\vec R}_+ \times \vec B) + \vec R_- \times (-q\dot{\vec R}_- \times \vec B) =$$

$$-q\vec B\left(\vec R_+\cdot\dot{\vec R}_+ - \vec R_-\cdot\dot{\vec R}_-\right) = -\frac{q\vec B}{2}\frac{d}{dt}\left(R_+^2 - R_-^2\right).$$

After integration, we find one more conservation law (conservation of the "generalized angular momentum"):

$$\vec L_O + \frac{q\vec B}{2}\left(R_+^2 - R_-^2\right) = \vec L_O + \frac{q\vec B}{2}\left((\vec R_+ + \vec R_-)\cdot(\vec R_+ - \vec R_-)\right)$$
$$= \vec L_O + \vec B(\vec R\cdot\vec p) = \text{const},$$

where $\vec R = \frac{1}{2}(\vec R_+ + \vec R_-)$ is the position of center of mass. We also used the fact that $q(\vec R_+ - \vec R_-) = 2q\vec r = \vec p$.

Initially, centre of mass coincides with origin ($\vec R_0 = 0$):

$$L_O(0) = I\omega_0 = 2m\frac{d^2}{4}2\frac{qB}{m} = qBd^2. \qquad (13)$$

At asymptote, the dipole has reversed direction $\vec p_1 = -\vec p_0$ and charges are travelling along parallel lines $x = D \pm r$ with the velocity $\vec v_1$:

$$L_O(\infty) + B(\vec R_1\cdot\vec p_1) = m(D-r)v_1 + m(D+r)v_1 - BDp_0$$
$$= 2mD\frac{p_0B}{m} - BDp_0 = BDp_0 = BDqd. \qquad (14)$$

Since (13) equals (14), we conclude that $D = d$.

We can arrive to the same conclusion differently. Notice that we are interested in the x coordinate of C at infinity:

$$D = x_\infty = \int_0^\infty v_x\,dt = \frac{qBd}{2m}\int_0^\infty \sin\theta\,dt.$$

From (12), we can express $\sin\theta$:

$$\int_0^\infty \sin\theta\,dt = -\frac{m^2}{q^2B^2}\int_0^\infty \ddot{\theta}\,dt = -\frac{m^2}{q^2B^2}(\dot{\theta}_1 - \dot{\theta}_0) = \frac{m^2}{q^2B^2}\omega_{\min} = \frac{2m}{qB}.$$

Finally,

$$D = \frac{qBd}{2m}\,\frac{2m}{qB} = d.$$

Note. If initial rotation is clockwise ($\omega_0 < 0$), the asymptote has an equation $x = -D$, but the distance to the origin remains the same.

Marking scheme

Part (a): Uniform linear motion		**pts**
a1	Rationalizes that the net force on the dipole is zero if the two poles move with equal velocities; Just argument $v = \text{const} \Rightarrow \sum\vec{F} = 0$ is 0 pts.	0.7
a2	Concludes that $\omega_0 = 0$.	0.3
a3	Using the argument of zero torque, concludes that the velocity should be perpendicular to the dipole; Just argument $\omega = \text{const} = 0 \Rightarrow \vec{\tau} = 0$: 0.4 pts	0.7
a4	States explicitly that $\vec{v}_0 \| Y$ (or $\perp X$).	0.3
	Total number of points for part (a)	**2.0**

Part (b): Circular motion		**pts**
b1	Derives expression for the magnitude of the net force on the dipole in terms of ω AND states explicitly that it is parallel to the dipole axis OR derives one single vector expression.	0.9
b2	Realizes (drawing or explicit statement) that $\vec{F}$ and the dipole axis point to the center of the orbit, and concludes that ω_0 is equal to the orbital angular velocity.	0.5
b3	Writes down Newton's second law for the circular motion.	0.5
b4	Makes use of the relation $v_0 = \lvert\omega\rvert R_c$.	0.2
b5	Derives expression for v_0 and specifies its direction (drawing or statement) OR derives one single vector expression for $\vec{v}_0$; if direction is wrong or missing 0.2 pts	0.3
b6	Derives explicitly $R_c = qbD/(2m\lvert\omega_0\rvert)$. If $\lvert\cdot\rvert$ is omitted, still full points.	0.3
b7	Writes down the coordinates of the center of the orbit; 0.2 for correct x_c (including sign), 0.1 for correct y_c; $x_c = qbD/(2m\omega_0)$ is a correct answer	0.3
	Total number of points for part (b)	**3.0**

Only one of the grading tables should be used for part (c), the one which results in a higher score.

Part (c): Reversal of the dipole		**pts**
c1	By integrating the equation(s) of motion derives a "generalized momentum" conservation law – a relationship between the linear momentum $2m\vec{v}$ and the dipole moment $\vec{p}$ – in vector form OR for the Cartesian components.	1.5
c2	States explicitly that the kinetic energy of the dipole conserves.	0.3
c3	Writes down explicit expression for the kinetic energy in terms of angular velocity and linear velocity of the center of mass.	0.5
c4	Realizes that ω_0 is minimal when $\omega_1 = 0$ in the reversed position.	0.2
c5	By using the "generalized momentum" conservation, derives explicit expression for the linear velocity v_1.	0.5
c6	Applies the conservation of energy to find relationship between v_1 and $\omega_{\min}$	0.8
c7	Derives the final expression for $\omega_{\min}$	0.2
	Total number of points for part (c)	**4.0**

Alternative approach: pendulum analogy

Part (c): Reversal of the dipole		**pts**
c1	Derives the expression $\tau = -B(\vec{p}\cdot\vec{v})$ for the torque. Even if the derivation has been made in parts (a) or (b), the points **should** be assigned to Task (c); If term $(\vec{B}\cdot\vec{p})$ is not cancelled, still full points	0.5
c2	By integrating the equations of motion, expresses v_x and v_y in terms of θ.	1.5
c3	Writes down the equation of rotational motion in terms of $\sin\theta$.	0.5
c4	States that the angular dynamics of the dipole is equivalent to a large-amplitude oscillation of a mathematical pendulum.	0.3
c5	Realizes that ω_0 is minimal when $\omega_1 = 0$ in the reversed position.	0.2
c6	Applies the conservation of energy to the "equivalent pendulum".	0.8
c7	Derives the final expression for $\omega_{\min}$	0.2
	Total number of points for part (c)	**4.0**

Part (d): Trajectory asymptote		**pts**
d1	Rationalizes that the asymptote is parallel to Y, i.e. $x = \pm D$.	0.1
d2	Rationalizes that asymptotically the motion is linear uniform	0.2
d3	Either finds conservation law $\vec{L}_O + \vec{B}(\vec{R}\cdot\vec{p})$ OR writes x_∞ as integral of v_x (with explicit expression for v_x) as a method to find D.	0.3
d4	Correctly computes generalized angular momentum at 0 and ∞ OR uses $\sin\theta \propto \ddot{\theta}$ in integral.	0.2
d5	Concludes that $D = d$.	0.2
	Total number of points for part (d)	**1.0**

E1: Colour and temperature

Theory

The infrared thermometer cannot be used to measure the filament temperature for several reasons – the range of the IR thermometer (stated on the instrument) only goes up to $500\,°C$. The filament is also too small to be the only thing measured. IR opacity of the glass bulb is also not guaranteed. Therefore, the only way to measure the temperature is indirectly through the colour index, for which the relation to temperature is provided.

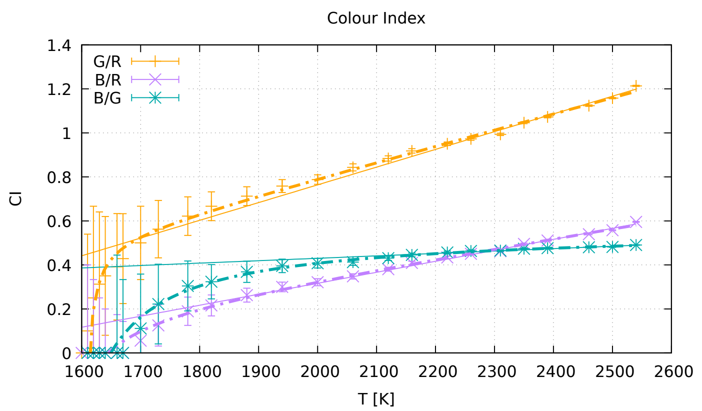
Wien's displacement law suggests that at lower temperatures, the light will contain more red component than green and blue, while at higher temperatures, the green and then the blue will increase faster than red, leading to increasing ratios G/R, B/R and B/G. We must, however, consider, which pair of filters will be the most suitable choice.

The values measured through different filters depend on the spectral response of each filter, including its overall opacity. It also depends on the sensitivity of the light meter to each wavelength. Instead of theoretical predictions, we are given reference measurements at known temperatures. If we plot the ratios for all three pairs, we observe that B/G is the least suitable, as it changes much less with temperature, compared to the other two. B/R and G/R are comparable, but the blue filter has lower transmittance, which will lead to lower accuracy (higher relative error).

Any pair of filters is a valid choice to proceed with the measurements, but will affect the end accuracy. Averaging the results is also an option, but including B/G combination may still reduce the accuracy of the end result.

To use the plot for converting the colour index to temperature, we need a trend line. A linear trend is enough to cover most of the range, except at lower temperatures, where the relationship tapers off. We can extend the range by combining two trends across the range, or to draw a smooth curve by hand. Zig-zag interpolation is less suitable due to scatter in the reference measurements.

Figure 1: Three choices of the colour index based on each pair of filters. Linear fits for the top part of the range are shown, and a smooth curve.



Using the absolute values from the table instead of ratios is not correct, as the intrinsic luminous flux of the light source and the measurement distance are not given.

Experiment

For the measurement of the power dependence of the temperature, we will read out the voltage and current from the power supply. To sample the expected curve of the $T(P)$ relationship, we must sample it sufficiently well, especially at lower powers where the temperature changes more quickly. We suggest sampling at least 8 powers/temperatures to cover the relationship more precisely and distinguish outliers from reliable measurements. For each power setting, we must measure the illuminance through the chosen filters by covering the sensor of the light meter with a filter. Covering the light meter filters *all* the light, including the light reflected from the walls and the floor, leading to a better measurement. Placing the filter next to the light source also introduces the risk of burning the filter. Planning ahead, we can simultaneously measure the illuminances without a filter, needed in Task 2.

Each colour index is then converted to a temperature by reading out from the calibration graph. We can also estimate the relationship by employing the Stefan-Boltzman law if we neglect other losses and the contribution of the ambient temperature:

$$T \propto \sqrt[4]{P}. \tag{1}$$

According to measurements with multiple light bulbs in different environments, the fit is

$$T = (1220\,\mathrm{KW}^{-1/4} \pm 20\,\mathrm{KW}^{-1/4})\sqrt[4]{P}, \tag{2}$$

which is used as a baseline for determining the RMS of students' measurements.

The *background* illuminance must be measured through all filters – it is most likely zero, but a good experimentalist *must* check, and if significant, it must be subtracted from measurements. This is also a way for us to detect if they left their desk lamp on – if the background differs significantly from the rest of the contestants.

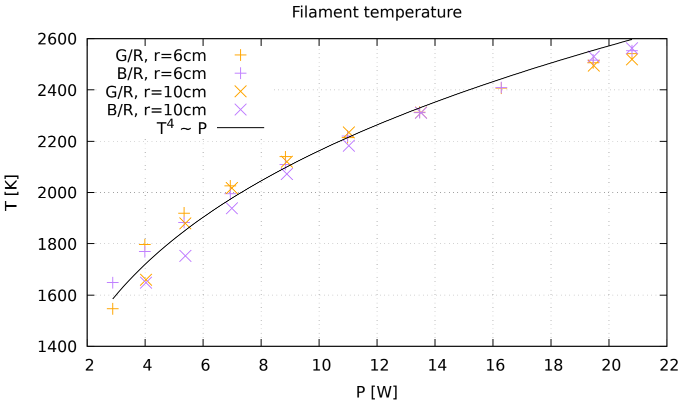
The distance between the light source and the light meter should be short enough to enable accurate measurements at lower powers. Distance can also be different for different power ranges, but care must be taken, as the effect of the finite size of the filament may play a role, as well as the changing reflections from surroundings and from the top of the light stand.

Marking scheme

General guidelines for marking in all tasks

- Granularity for marks is 0.1 p.
- measurements/results given with inappropriate number of significant figures may get deducted 0.1 p. (rule of no propagating error applies)
- A simple numerical error resulting from a typo is punished by 0.1 p unless the grading scheme explicitly says otherwise.

Figure 2: Dependence of the temperature on the power with a superimposed prediction from the Stefan-Boltzman law. Shown is a comparison measured at two distances and using two combinations of filters. We see the trends are comparable and remain within $\pm 100\,\mathrm{K}$, and RMS is around $\pm 40\,\mathrm{K}$ based on multiple measurements.



- Errors in theoretical derivations which cause dimensionally wrong results are punished by at least 50% of the marks for the derivation unless the grading scheme explicitly says otherwise.
- Propagating errors are not punished repeatedly unless they either lead to considerable simplifications or wrong results whose validity can easily be checked later.
- Negative points cannot decrease the score under the same section (A1,A2,...) below zero.

Calibration		**Points**
A1	**Plotting**	**1.0**
	Compute color indices for sufficient number of data points over range	0.3
	Plot of color indices	0.5
	Proper axis labels & ticks	0.1
	Sufficient size of graph for precise readout ($\geq \frac{1}{2}$ page)	0.1
	Each point computed/drawn incorrectly	-0.1
A2	**Trend line**	**0.5**
	Smooth trend curve or a composite of linear trends	0.5
	Single linear trend line (generates outliers at some temperatures)	or 0.3
	Zig zag connected points (=used interpolation for readout), or pointwise connected curve	or 0.1
Total on Calibration		**1.5**

This section is only for the calibration data (tables, graphs) based on the given table. Plots made based on IR measurements, get zero points.

Full points are given for a single colour index graph, or for multiple colour indices which may be on the same plot or on the separate plots.

Absolute values plotted: At most 0.2 points for A1 (if axes and ticks are done correctly, see the table above), if they plotted the absolute values through one filter or differences of values of several filters, instead of colour indices. The illuminance depends on the distance and the brightness of the light source, so absolute values cannot be used to determine the temperature. Points for A2 may nevertheless be given.

Measurements		**Points**
B1	**Data collection †**	**1.2**
	Measured U, I, E through 2 or 3 filters at $n \leq 8$ points (for 1 filter only $3/4n$ and rounded, for measurement of U and I or RG(B) only, which typically happens with measurements with IR thermometer, max. 0.3)	$0.1n$
	Compute P	0.1
	Compute color indices or convert directly to temperature (from graph or trend in A2)	0.1
	No points below $5\,\mathrm{W}$	-0.1
	No points above $16\,\mathrm{W}$	-0.1
	Measured RGB background	0.1
	Determined background constancy (e.g. measured at the beginning and the end)	0.1
B2	**Temperature plot ‡**	**0.9**
	Determine temperatures	0.3
	Plot of data	0.3
	Add best fit curve	0.1
	Proper axis labels & ticks	0.1
	Proper size of graph	0.1
	Each point determined/drawn incorrectly	-0.1
	Used G/B index	-0.2
B3	**Result quality ‡**	**0.4**
	Nonlinearity of the relationship is visible	0.1
	Nonmonotonous relationship	-0.1
	RMS from best fit within $40\,\mathrm{K}$	0.3
	RMS from best fit within $80\,\mathrm{K}$	or 0.2
	RMS from best fit within $120\,\mathrm{K}$	or 0.1
Total on Measurements		**2.5**

† **Presentation of data:** If U and I are directly multiplied and only P values are presented, no marks are deducted.

Background: Points for background are only granted if the background is subtracted from the measurements, or can be reasonably neglected.

‡ **IR measurement:** If temperatures are "determined" from IR measurements or any other method unfit to determine the temperature, no marks are given for B2. The same holds for B3, because presence of nonlinearity is not an indicator of quality for meaningless data.

To determine the RMS at B3, we compare it to Eq. 2 and take the root of the mean squared deviation. Reasonably exclude any outlying measurements at very low powers where we expect large deviations. The RMS calculations can be handled by the auxiliary Excel file.

E2: Efficacy

Theory

Light sources do not radiate in all directions equally. The angular distribution of luminous flux Φ (luminous intensity) must be integrated over the solid angle. A light meter at distance r to the light source, oriented so that the light falls on it perpendicularly, measures the illuminance E of a certain part of the imagined integration sphere surrounding the light source:

$$\Phi = \oint E(\Omega) r^2 \mathrm{d}\Omega. \tag{3}$$

The LED only shines the light into a hemisphere, and has cylindrical symmetry around the direction straight ahead, so we can simplify the expression,

$$\Phi_{LED} = 2\pi \int_0^{\pi/2} E(\theta) r^2 \sin\theta \, \mathrm{d}\theta, \tag{4}$$

and for the incandescent bulb, the symmetry axis is perpendicular to the direction straight ahead, and shines into full solid angle:

$$\Phi_W = 4\pi \int_0^{\pi/2} E(\theta) r^2 \cos\theta \, \mathrm{d}\theta. \tag{5}$$

The integrals will have to be evaluated numerically – it can be done by using the trapezoidal or the Simpson method, or by using the formula for a spherical segment area given in the hint:

$$\Phi = 2\pi r^2 \sum_i E(\theta_{i+1/2})(\cos\theta_i - \cos\theta_{i+1}) \tag{6}$$

and equivalent (but with sin $\iff$ cos) for the incandescent bulb. Here, choosing evaluation points in the middles of intervals is better than choosing one of the edge points. However, the exception are the “edge” measurements, where the measurement point is actually in the middle of the interval – the point straight ahead for the LED is in the middle of the spherical cap. The same goes for the “poles” of the incandescent light bulb.

The ratio between the head-on measured illuminance and the luminous flux, can be expressed as

$$\Phi = C r^2 E(0), \tag{7}$$

or, more intuitively, as a correction factor to the isotropic source:

$$\Phi = \{4\pi, 2\pi\} \tilde{C} r^2 E(0). \tag{8}$$

Analytical estimates

One possible pathway is to estimate these factors without measurements, using reasonable assumptions about the light distribution. The LED can be assumed a planar emitter, with a cosine distribution of luminous flux:

$$\tilde{C}_{LED} = \frac{\int_0^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta}{\int_0^{\pi/2} \sin\theta \, \mathrm{d}\theta} = \frac{1}{2} \tag{9}$$

which turns out to match the experiment well.

For the incandescent bulb, a similar assumption can be made based on a thin filament model. The different orientation of the symmetry axis leads to a different result:

$$\tilde{C}_W = \frac{\int_0^{\pi/2} \cos^2\theta \, \mathrm{d}\theta}{\int_0^{\pi/2} \cos\theta \, \mathrm{d}\theta} = \frac{\pi}{4} \approx 0.79. \tag{10}$$

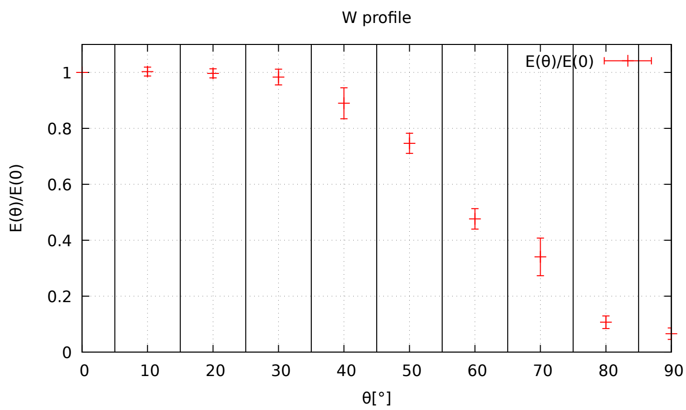
These approximations can be used to a good effect but are not required in the experimental task.

Experiment

To measure the angular dependence, a suitable distance to the light source must be chosen. Too far, and the signal becomes weak and any background could become noticeable. It is advisable to measure the angular dependence at the highest power in order to improve the signal to background ratio. Measurement can also be performed through one of the filters.

For the incandescent bulb, the finite size of the filament becomes an issue if we measure too close to the bulb. This becomes noticeable at distances lower than $10\,\mathrm{cm}$. This was not an issue for the colour index measurement, but it matters for the absolute flux estimation.

Figure 3: Angular profile of the incandescent light bulb, measured at $r = 15\,\mathrm{cm}$, $P = 20.6\,\mathrm{W}$ in increments $10°$. Vertical lines are the division angles for formula (6). We obtain $C = 10.01$ ($\tilde{C} = 0.80$).

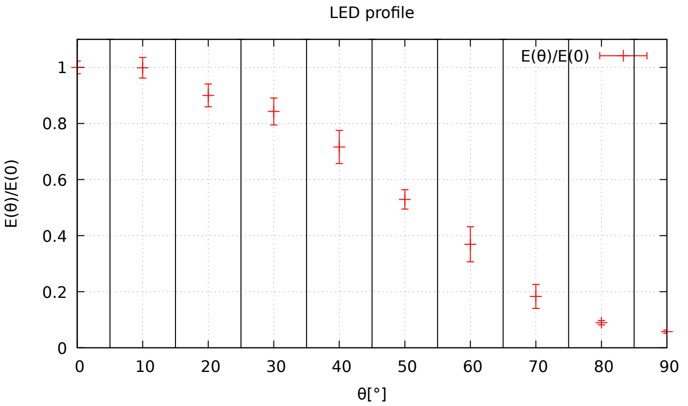


To describe the inflection point in the light distribution well, we will need at least 5 measurements in the $\theta \in [0, \pi/2]$ interval. We can either rotate the light source on the spot, or position the light meter at different angles in relation to the stationary light source.

For the light distribution left-right symmetry can be assumed, or, alternatively, the entire $\theta \in [-90°, 90°]$ range can be measured, allowing to take into account asymmetries and an angular offset in the light distribution. The straight ahead measurement is centered in a symmetric band, which needs care so it is not double-counted in case only half of the range is integrated and then doubled.

With the conversion factors known, the luminous efficacy can be determined by measuring the frontal illuminance at powers that cover the entire range from the lowest detectable illuminance to the maximum allowed

Figure 4: Angular profile of the LED, measured at $r = 10\,\text{cm}$, $P = 1.33\,\text{W}$ in increments $10°$. Vertical lines are the division angles for formula (6). We obtain $C = 2.63$ ($\tilde{C} = 0.42$).



	C	$\tilde{C}$
W	10.01	0.80
LED	2.63	0.42

Table 1: Example values of the conversion factor between the frontally measured illuminance and the luminous flux for both light sources. The values will vary within some wider distribution because of varying light sources and other errors, which is indicated by the brackets in the grading sheet.

power. For the incandescent bulb, this measurement can be done simultaneously with Task 1 for better time efficiency.

It is not required to measure at the same distance as the angular dependence. Multiple distances may also be used.

We have to avoid placing any additional objects near the light source to avoid introducing more reflected or blocked light – such as placing the light source directly on the white paper, or having other obstructions such as the black paper screen or any filters too close to the light bulb.

To plot the efficacy, we divide the Φ obtained from eq. (3) for each of the light sources, with $P = UI$ read out from the power supply.

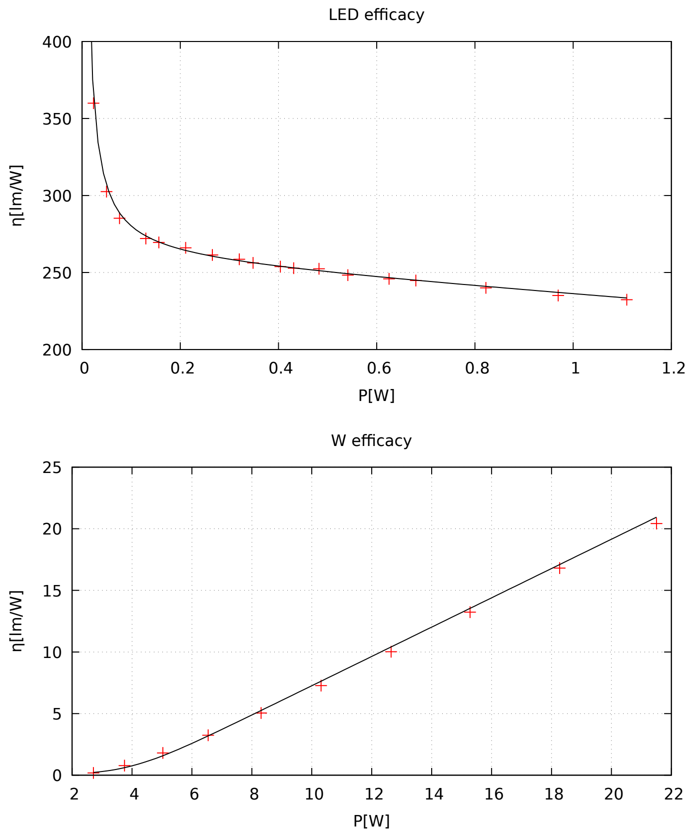
The result shows that the efficacy of the incandescent light starts out at zero at low powers and increasing with power, as its temperature increases. The LED has the highest efficacy at lowest powers, then it drops off at higher powers, mostly due to increased temperature of the light emitting junction.

At the lowest settable currents, the readout on the power source is no longer reliable – for example, LED may glow slightly even at $0\,\text{A}$. The pole at the origin can be attributed to this source of error.

Marking scheme

The basic equations could be stated in a separate section of the solution, or spread over different parts of the solution.

Figure 5: Efficacy of both light sources depending on the input power.



Theoretical background		**Points**
A1	**Eq. (3) or equivalent**	**0.5**
	Dependence on r^{-2}	0.2
	Dependence on angle (noticing anisotropy)	0.3
A2	Take into account cylindrical symmetry (each)	2×0.1
	Choose right symmetry axis for light sources (each)	2×0.1
	Correct factors of 2π and 4π (LED and bulb, respectively)	0.1
	Assume area of the sensor comes into calculations	-0.2
Total on Theory		**1.0**

$2\times$ means one for each light source (bulb, LED).

Angular measurement		Points
B1	**Incandescent measurement**	**0.9**
	Measured at $n \leq 6$ or more angles between 0 and $\pi/2$	$0.1n$
	Specified auxiliary data (power, distance)	0.2
	Measured full angular range $[-90°, 90°]$ (e.g. by rotating light)	0.1
	Maximum illuminance below 100 lx	-0.2
	Measured closer than 10 cm	-0.1
B2	**Integration procedure**	**0.7**
	Used the hint	0.7
	↳Values at the edges of intervals	-0.1
	↳Double counting of the equator	-0.1
	Used trapezoidal or similar rule	or 0.7
	↳Incorrect treatment of edge values	-0.1
	↳Double counting of the equator	-0.1
	Analytically derived eq. (10) or similar	or 0.7
	Averaged values without weights	or 0.3
B3	**C result accuracy for Tungsten**	**0.3**
	Calculated C or equivalent	0.1
	Value of $C \in [9.7, 10.3]$	0.2
	Value of $C \in [9.4, 10.6]$	or 0.1
B4	**LED angular measurement**	**0.9**
	Measured at $n \leq 6$ or more angles between 0 and $\pi/2$	$0.1n$
	Specified auxiliary data (power, distance)	0.2
	Measured full angular range $[-90°, 90°]$ (e.g. by rotating light)	0.1
	Maximum illuminance below 100 lx	-0.2
B5	**Integration same as B2**	**0.7**
B6	**C result accuracy for LED**	**0.3**
	Calculated C or equivalent	0.1
	Value of $C \in [2.8, 3.2]$	0.2
	Value of $C \in [2.6, 3.4]$	or 0.1
B7	Background measured	0.1
	Background constancy check (multiple measurements)	0.1
Total on Angular		**4.0**

Plotting the angular dependence is not required for the procedure, but it counts as 0.2 points if the integration procedure was otherwise not performed.

If a matching analytical value for both conversions are obtained, but angular dependence is not measured, the contestant can get all marks except for the measurement (1.0 out of 1.9 for each light source).

If a comparison of analytical and experimental estimate are done, or if an analytical model is fitted to the experimental data, the procedure is correct, and up to additional $+0.3$ points per light source may be given to compensate points lost for steps that merit points that may not be necessary for the method used.

Efficacy measurement		Points
C1	**Incandescent measurement**	**0.7**
	Measured U, I, E for incandescent light at $n \leq 7$ points	$0.1n$
	No measurements above 16 W	-0.1
C2	**Plotting bulb efficacy**	**0.8**
	Convert and plot points $n \leq 6$ (if converted into flux, not efficacy max 0.3)	$0.1n$
	Missing axis labels	-0.1
	Deviation from a monotonous shape	-0.1
†	Values within RMS< 0.2	0.2
	Values within RMS< 0.5	or 0.1
C3	**LED measurement**	**0.7**
	Measured U, I, E for LED at $n \leq 7$ points	$0.1n$
	Fewer than 2 measurements above 0.2 W	-0.2
	Fewer than 2 measurements below 0.2 W	-0.1
C4	**Plotting LED efficacy**	**0.8**
	Convert and plot points $n \leq 6$ (if converted into flux, not efficacy max 0.3)	$0.1n$
	Missing axis labels	-0.1
	Deviation from a monotonous shape	-0.1
‡	Values within RMS< 5	0.2
	Values within RMS< 10	or 0.1
Total on Efficacy		**3.0**

† Vary the C factor within the range $[9.4, 10.6]$ to check for a better match. This helps remove propagation error from a badly determined C and additionally, any small variations in power output between light bulbs.

‡ Vary the C factor within the range $[2.6, 3.4]$ to check if there is a better match. The LEDs may vary in light distribution and absolute intensity, so the trend cannot be matched 1:1. Exclude also the lowest power outliers from the assessment, as the singularity can give a high RMS without being significant to the LED itself. It depends on the voltage bias of the power source and resistivity of junctions.

RMS comparison: To make the comparison of the graphed solutions to the reference less subjective, we do a root mean square comparison: average square deviation from an empirical model curve based on a larger number of measurements in a controlled environment (Fig. 5). For the light bulb efficacy, we use

$$\eta(P) = \ln(1 + \exp(1.189P - 4.632)) \tag{11}$$

which is just a linear function smoothly flattened at the bottom.

For the LED, we use

$$\eta(P) = 2.56/P - 23.78P + 259.56 \tag{12}$$

where possible intensity variations can be compensated by allowing C variation in C2/C4. Variations in the $1/P$ part (due to different offsets in power supply readout) can be compensated by excluding the low-power measurements from the RMS calculations in a reasonable way.

All these are in base units without prefixes (omitted for clarity).

If angular dependence is ignored use this table as a shortcut for grading

A1	dependence on r^{-2} only	0.2
A2	Correct factor of 2π or 4π only	0.1
B1-B6	not applicable	0.0
B7	Background measured	0.1
	Background constancy check (multiple measurements)	0.1
C1	No change	0.7
C2	Rescale with correct C to check RMS	0.8
C3	No change	0.7
C4	Rescale with correct C to check RMS	0.8

E3: Radiative heating

Theory

The plate receives a radiant flux density j, determined by the power P of the light source, and the distance r between the target and the light source. The light source does not shine equal amounts of light in all directions, therefore we must use the correction factor C, derived in Task 2, to convert from the total radiant flux to forward radiant flux density.

$$P = Cr^2 j \to j = \frac{P}{Cr^2}. \tag{13}$$

Not necessary, but also correct, is to (numerically) integrate/average across the entire plate, $j(\pi r^2) = P\int Cr^{-2}\cos\theta \, \mathrm{d}A$ to take into account spatial variation of C, r and θ (angle of incidence).

The incident flux density is dissipated to the environment directly, as well as by heat conduction through the plate. Mark by T_F the front temperature and T_B the back temperature. Conservation of energy gives us the system of equations

$$j = h(T_F - T_0) + \frac{\lambda}{d}(T_F - T_B) \tag{14}$$

$$0 = h(T_B - T_0) + \frac{\lambda}{d}(T_B - T_F). \tag{15}$$

This system of equations leads to the following relations:

$$j = h(T_F + T_B - 2T_0) \tag{16}$$

$$j = (h + 2\frac{\lambda}{d})(T_F - T_B). \tag{17}$$

Any linear combination of equations (14,15) also allows determination of both h and λ. A particular linear combination that may be used is the isolation of individual temperatures:

$$T_F - T_0 = \frac{1}{2}\left(\frac{1}{h} + \frac{1}{h + 2\frac{\lambda}{d}}\right) j \tag{18}$$

$$T_B - T_0 = \frac{1}{2}\left(\frac{1}{h} - \frac{1}{h + 2\frac{\lambda}{d}}\right) j. \tag{19}$$

In our system, $2\frac{\lambda}{d} > h$, but still in the same order of magnitude. Treating the slope of T_F as $1/(2h)$ or the slope of $T_F - T_B$ as $(2\lambda/d)^{-1}$ is a reasonable approximation, but still not theoretically correct.

Error analysis

Errors should be propagated from the slope. For example, if they obtain slopes $k_1 = 1/h$ and $k_2 = 1/(h + 2\lambda/d)$, they should propagate the errors. We should allow both straight addition of error contributions of different terms, or adding squared errors (independent errors), e.g.

$$h = \frac{1}{k_1} \pm \frac{\sigma_1}{k_1^2} \tag{20}$$

$$\lambda = \tfrac{d}{2}\left(\frac{1}{k_2} - \frac{1}{k_1}\right) \pm \tfrac{d}{2}\left(\frac{\sigma_1}{k_1^2} + \frac{\sigma_2}{k_2^2}\right) \tag{21}$$

and analogously for other slope definitions.

Albedo

For the white plate, only a part of the incident flux is absorbed, so we replace j by $j(1-a)$ if a is the albedo:

$$j = (1-a)\frac{P}{Cr^2}. \tag{22}$$

As a consequence, any slope measured for both plates will be in the ratio $(1-a)$ to each other. This can be expressed as a fraction of trend slopes, ratio of temperature differences, or similar.

Experiment

The radiant flux density can be varied in two main ways, or a combination of both: by changing the distance, or by changing the current through the light bulb. Both methods are acceptable, but varying the current also changes the spectrum and the efficiency of the light bulb, so it may produce biased and nonlinear results. The students should know that varying a single parameter is the correct procedure.

The required measurements in this task are the front and back temperature at different powers, for black and white plate. It is essential to wait for equilibration, which includes waiting the back temperature to stabilize. It is advisable to measure starting with the lowest flux density, because it will require the least equilibration time from the initial room temperature of the plate.

The target should not be too close to the light source, not only because of the risk of burning, but also because close to the light bulb, the light is very nonuniformly distributed across the plate. Increased convection rate due to high temperature also starts deviating from the linear regime. Placing the target too far from the light source leads to a negligible heating and thus a very large relative error in temperature differences, especially for the white plate.

In this task, the measurements are subject to many sources of errors: measuring from different distances and at different angles may include different proportions of background or reflected IR radiation from the light source (if the targeted area is still illuminated), if the measurement takes too long, the plate may start cooling down (this is noticeable in a few seconds), air currents may increase convective heat dissipation, and the ambient temperature may also change during the measurement (especially if the light source is placed too close to the wall, or if the power source's fan exhaust is too close to the measurement setup). The errors are most noticeable at low radiant flux and for the white plate, where increases in temperature are the smallest.

For these reasons, it is advisable to take more than one measurement per data point and average the results, and to cover a sufficiently wide range to reduce the slope error. At least 3 points are needed to draw a trend, but 5 is better. With more points, it is easier to spot outliers and utilise the measurements which are least subjected to errors. Back and front temperatures are best measured in pairs one after the other to reduce the error in the temperature difference signal due to changing conditions.

Note: the ambient temperature T_0 is an effective temperature that combines air temperature and radiative exchange with the surrounding walls, ceiling and other objects. We do not need its value, we only need the slopes of the linear trends. Inexact T_0 can lead to inaccuracies if used together with an assumption the linear relations go through the origin. T_0 cannot be reliably determined by measuring surrounding temperatures, but it can be estimated by measuring the equilibrium temperature of the plate in the absence of the light source.

The measurements of the front and back temperature at different radiant fluxes, must be processed and plotted to extract the necessary slopes. For the black plate, two plots will be needed, based on equations (14,15), equations (16,17), or any linearly independent pair. Linear regression gives us the slopes h and $h + 2\frac{\lambda}{d}$ (or their reciprocals). T_0 is best determined by the $j = 0$ intercept of the trend line for (Eq. 16) or any equivalent plot, and should match T_0 determined by other methods. If measured correctly, the intercept of the trend line for (Eq. 17) should be zero within the error margin.

It is possible to calculate the necessary slopes from a measurement at a single input power (for each plate color), if T_0 is measured well. This can be done without a graph. However, using multiple measurements decreases the impact of statistical errors and enables us to better estimate the error, so a single measurement will carry a significant error.

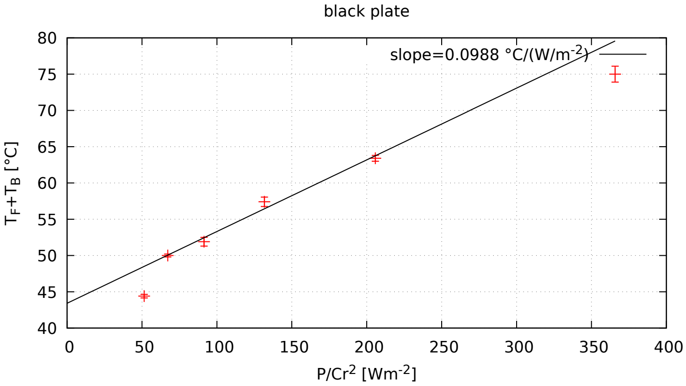
Albedo, as defined in the task text through *irradiance* units, cannot be measured using a light meter, which measures in *photometric* units. Additionally, light reflected from a white plate introduces additional geometric considerations and angular distribution of reflected light, that cannot easily be taken into account.

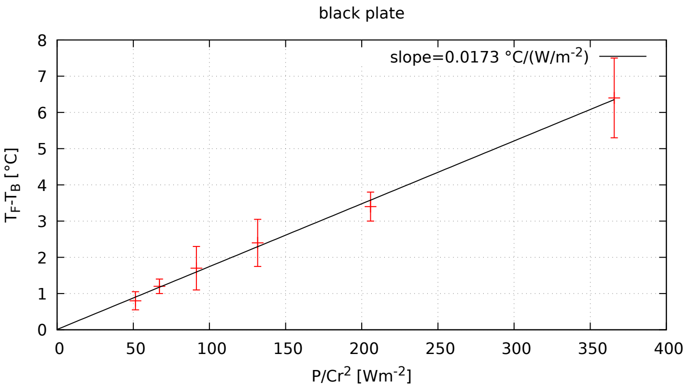
The albedo can be estimated as a fraction of the corresponding line slopes between the black and the white plate, taking any of the relations (14,15,16,17). This means that for the white plate, measuring only one side of the plate is enough to determine the albedo, assuming h and λ remain the same. The difference slope or the back temperature slope are the least suitable, as they introduce a large relative error to the measurement due to a minimal increase in temperature.

Marking scheme

The basic equations could be stated in a separate section of the solution, or spread over different parts of the solution.

Figure 6: Black plate measurements for eqs. (16,17). The flux density j was varied by changing the distance r. The slope of the first graph equals h^{-1}, so $h = 10\,\mathrm{Wm^{-2}K^{-1}}$. Lowest and highest measurement were excluded from the fit. The intercept is $2T_0$.
The slope of the second graph equals $(h + 2\lambda/d)^{-1}$, so $\lambda = 0.072\,\mathrm{Wm^{-1}K^{-1}}$. The intercept is reasonably close to 0.



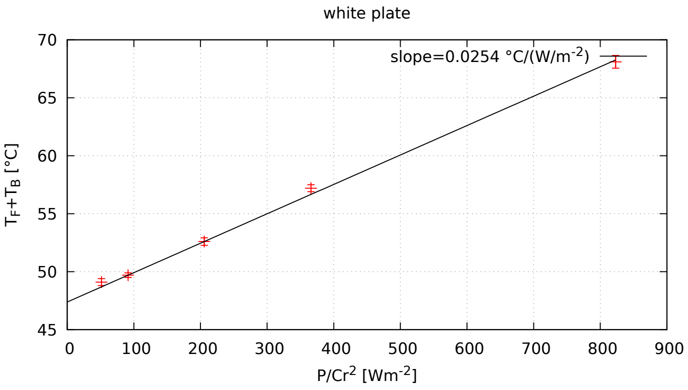


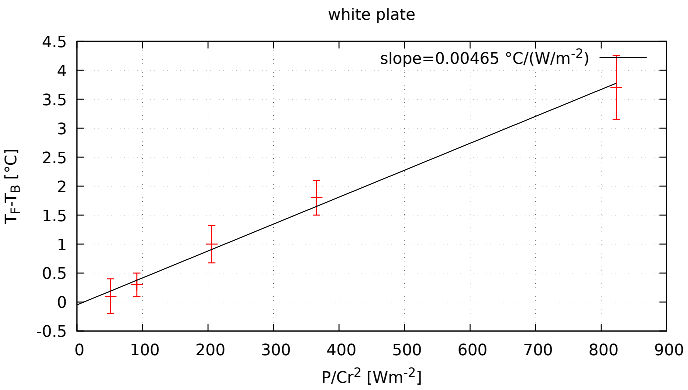
Theoretical background		Points
A1	**Power to irradiance**	**0.4**
	Correct eq. (13) or equivalent	0.3
	Realizing the same geometry from Task 2 applies (C or equivalent)	0.1
A2	**Heat exchange balance**	**0.6**
	Correct **each** of eqs. (14,15) or equivalent.	2×0.3
	Each partially correct eq., e.g. assumed $T_B = T_0$ in (14) or missing 2 or h in (17)	or 0.1
A3	**Albedo balance**	**0.2**
	Correct eq. (22) or equivalent	0.2
Total on Theory		**1.2**

The assumption that the left hand side of equation $P/A = h(T - T_0)$ distributes the *full* power of the light source to the area of the plate, indicates a lack of understanding and merits 0 points for theory part.

If conduction is not considered at all a maximum of 0.1 points is given to A2.

Figure 7: White plate measurements for eqs. (16,17). The slope ratio with the black plate result is $(1-a) = 0.0254/0.0988 = 0.26$ for the first graph. The second graph confirms this with a closely matching $(1-a) = 0.00465/0.0173 = 0.27$.





Measurements		Points
B1	$n \le 5$ measurements of T_F (black)	$0.1n$
B2	$n \le 5$ measurements of T_B (black)	$0.1n$
B3	$n \le 5$ measurements of T_F, T_B or both (white)	$0.1n$
B4	Measured the unchanging values (distance if U, I varied, U, I if distance varied)	0.2
	Estimated measurement errors (at least separate for each plate color) – the instrument precision is not a valid error estimate	0.2
	Estimated measurement errors (common for all)	or 0.1
	Measured by varying the current (not distance)	-0.2
Total on Dissipation		**1.9**

Determination of h and λ will require extraction of two trend lines from two plots. Plotting on the same graph counts as two, but the vertical axes must be labelled correctly. The trend lines will have a $j = 0$ intercept that will be 0 in case of temperature difference, and related to ambient temperature otherwise. Using r^{-2} or P instead of j as an axis is valid as long as the conversion is done correctly at the slope readout.

Dissipation parameters		Points
C1	**First of the two plots**	**0.8**
	$n \le 5$ correctly converted and drawn points	$0.1n$
	Correct trend line	0.1
	Correct slope readout	0.1
	Slope error estimate	0.1
	Intercept disagrees with expectations	-0.1
	Missing axis labels, ticks or unsuitable size	-0.1
C2	**Second of the two plots**	**0.8**
	Same breakdown as C1	
C3	**Calculation of h**	**0.8**
	Correct algebraic relation to slopes	0.2
	Numerical value within $[10, 14]$	0.3
	Numerical value within $[8, 16]$	or 0.2
	Numerical value within $[6, 18]$	or 0.1
	Correct error analysis	0.2
	Error estimate < 1 (if error analysis is reasonable)	0.1
C4	**Calculation of λ**	**0.8**
	Correct algebraic relation to slopes	0.2
	Numerical value within $[0.06, 0.08]$	0.3
	Numerical value within $[0.05, 0.09]$	or 0.2
	Numerical value within $[0.04, 0.10]$	or 0.1
	Correct error analysis	0.2
	Error estimate < 0.01 (if error analysis is reasonable)	0.1
Total on Dissipation		**3.2**

The values are in SI base units.

Without plotting: If the entire fitting process is done numerically without plotting, use equivalent concepts to the grading above – tables instead of plots, slope calculations instead of trend lines, etc. As the plot is not required, a correct procedure can yield full points.

The error analysis in this case may consist of doing the entire procedure (e.g. using a single measurement with T_0 knowledge, or two points without background), with multiple measurement runs and doing statistics. Another option is propagating relative errors from the single measurement errors. The main criterion is, that the error source is statistical, not instrumental.

The point count includes the origin for the plot of the temperature difference (eq. 17).

The slope error is the main source of error – distances and powers can be considered accurate. Error estimate on the slope can be done based on point scatter (but not with fewer than 5 points), with or without taking into account errorbars (if the students estimated them).

Error propagation: if a wrong value of C is used, recalculate with a suitable value and grade accordingly.

Albedo		Points
D1	**One or more plots**	**1.0**
	$n \leq 5$ correctly converted and drawn points	$0.1n$
	Estimated individual measurement error	0.1
	Correct trend line(s)	0.2
	Correct slope readout	0.1
	Slope error estimate	0.1
	Intercept disagrees with expectations	-0.1
	Missing axis labels	-0.1
D2	**Data processing**	**0.7**
	Correct algebraic expression for a	0.2
	Numerical value $a \in [0.65, 0.75]$	0.2
	Numerical value $a \in [0.6, 0.8]$	or 0.1
	Correct error analysis	0.2
	Error estimate < 0.05 (if error analysis is reasonable)	0.1
Total on Albedo		**1.7**

The possibility of measuring both temperatures allows combinations where both sets of data can be used for albedo estimation – by averaging two slope ratios, or similar. This is also a valid approach.

In cases where only pointwise numerical evaluation using several data points is employed a maximum of 0.5 points for D1 (0.3 conversion of data, 0.2 for error estimates) and a maximum of 0.7 for D2 will be awarded. For evaluation with one data point only a maximum of 0.2 points for D1 (0.1 for conversion of data, 0.1 for error estimate) and 0.5 for D2 (0.2 algebraic expression, 0.2 value, 0.1 error analysis) will be awarded.