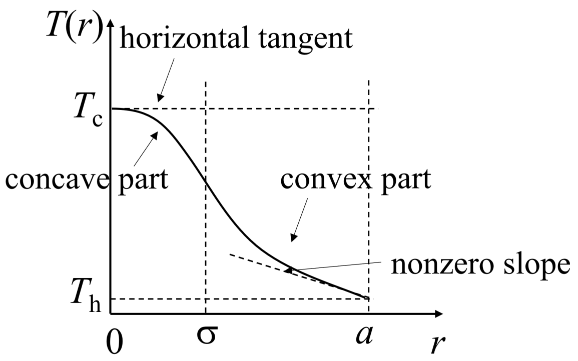
T1: Thermal lens - Solution

(a) Drawing a $T(r)$ graph

The graph should present or clearly infer for the four elements shown in the figure.



Element of the graph	Pts.
Horizontal tangent at $r = 0$	0.5
Graph is concave and decreasing in $0 \le r \le \sigma$	0.5
Graph is convex and decreasing in $\sigma \le r \le a$	0.5
Nonzero negative slope at $r = a$	0.5
Totally on (a)	**2.0**

(b) Finding T_c

Approach with a direct solution of the heat-transport equation.

Consider a cylindrical cut of the disk with radius r. Let $P_{abs}(r)$ be the portion of laser power absorbed within the cylinder. The absorbed power is being transferred as heat towards the outer holder through the circumvent surface $2\pi br$ of the cylinder. The heat-transport equation reads:

$$-k(2\pi rb)\frac{\mathrm{d}T(r)}{\mathrm{d}r} = P_{abs}(r) \tag{1}$$

Depending on r, the absorbed power is given by two different expressions. Within the illuminated area $0 \le r \le \sigma$, the incident light intensity $I = P_L/(\pi\sigma^2)$ is constant, so the absorbed power is:

$$P_{abs}(r) = I\pi r^2 = AP_L\frac{r^2}{\sigma^2} \tag{2}$$

and the solution for $T(r)$ is quadratic in r:

$$T(r) = T_c - \frac{AP_L r^2}{4\pi kb\sigma^2} \tag{3}$$

where $T_c = T(0)$ is the temperature at the center. It is clear from that expression that the parameter m is:

$$m = -\frac{AP_L}{4\pi kb\sigma^2} = -1.1 \cdot 10^7\,\mathrm{K m^{-2}} \tag{4}$$

Outside the illuminated area, i.e. for $\sigma \le r \le a$, $P_{abs}(r) = AP_L$, which does not depend on r. The solution for $T(r)$ is logarithmic in r:

$$T(r) = T_h + \frac{AP_L}{2\pi kb}\ln\left(\frac{a}{r}\right) \tag{5}$$

where $T_h = T(a)$ is the temperature of the holder, which is equal to the temperature along the outer rim of the disk. After matching the two solutions at $r = \sigma$ we obtain:

$$T_c = T_h + \frac{AP_L}{4\pi kb}\left[1 + 2\ln\left(\frac{a}{\sigma}\right)\right] = 41^\circ\mathrm{C} \tag{6}$$

Task	Pts.
Writes down the heat-transport equation in radial coordinates	0.7
Derives an expression for P_{abs} at $0 \le r \le \sigma$	0.5
Finds a quadratic solution for $T(r)$ in the region $0 \le r \le \sigma$ (Subtract 0.2 pts. if he boundary condition $T(0) = T_c$ has not been accounted for)	0.5
Identifies the expression for m (Subtract 0.1 pts. if the minus sign is missing)	0.2
Calculates m numerically (Subtract 0.1 pts. if the minus sign is missing)	0.2
Derives an expression for P_{abs} at $\sigma \le r \le a$	0.3
Finds a logarithmic solution for $T(r)$ in the region $\sigma \le r < a$ (Subtract 0.2 pts. if he boundary condition $T(a) = T_h$ has not been accounted for)	0.5
Sets up an equation by matching the two solutions at $r = a$	0.6
Writes down the final expression for T_c	0.2
Calculates T_c numerically	0.3
Totally on (b)	**4.0**

Alternative approach – direct piece-wise integration of the heat-transport equation.

After realizing that $P_{abs}(r)$ is given by a piece-wise function:

$$P_{abs} = \begin{cases} AP_L r^2/\sigma^2 & \text{if } 0 \le r \le \sigma \\ AP_L = \text{const} & \text{if } \sigma \le r \le a \end{cases} \tag{7}$$

the student may substitute the given solution $T(r) = T_c + mr^2$ into heat-transport equation (1) for $0 \le r \le \sigma$. This gives directly the expression (4) for the parameter m. The parameter T_c could be easily identified with the temperature $T(0)$ at the center of the disk. On the other hand, $T_h = T(a)$ due to the thermal contact between the rim of the disk and the holder. It follows from the heat-transport equation that:

$$-\frac{\mathrm{d}T(r)}{\mathrm{d}r} = \frac{P_{abs}(r)}{2\pi kbr} \tag{8}$$

The piece-wise integration of the two sides of the equation in the interval $0 \le r \le a$ gives:

$$\begin{aligned} T(0) - T(a) = T_c - T_h &= \int_0^a \frac{P_{abs}(r)}{2\pi kbr}\mathrm{d}r \\ &= \int_0^\sigma \frac{P_{abs}(r)}{2\pi kbr}\mathrm{d}r + \int_\sigma^a \frac{P_{abs}(r)}{2\pi kbr}\mathrm{d}r \\ &= \frac{AP_L}{4\pi kb}\left[1 + 2\ln\left(\frac{a}{\sigma}\right)\right] \end{aligned} \tag{9}$$

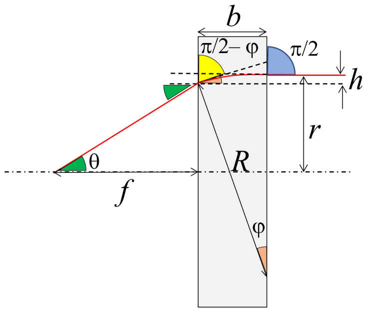
which is equivalent to the expression (6) for T_c.

Task	Pts.
Writes down the heat-transport equation in radial coordinates	0.7
Derives an expression for P_{abs} at $0 \le r \le \sigma$	0.5
Derives an expression for P_{abs} at $\sigma \le r \le a$	0.3
Substitutes the given form of $T(r)$ into heat-transport equation for the region $0 \le r \le \sigma$	0.3
Obtains the expression for m (Subtract 0.1 pts. if the minus sign is missing)	0.2
Calculates m numerically (Subtract 0.1 pts. if the minus sign is missing)	0.2
States that $T_c = T(0)$	0.2
States that $T_h = T(a)$	0.2
Expresses $T_c - T_h$ through integral of dT/dr in the $0 \le r \le a$ interval	0.3
Calculates the integral in the $0 \le r \le \sigma$ interval	0.3
Calculates the integral in the $\sigma \le r \le a$ interval	0.3
Writes down the final expression for T_c	0.2
Calculates T_c numerically	0.3
Totally on (b)	**4.0**

(c) Finding the focal length

Approach based on the Fermat's principle

We consider only the illuminated area of the disk ($0 \le r \le \sigma$). Due to the nonuniform temperature distribution, the index of refraction is also r-dependent, which leads to a bending of the light rays incident at nonzero radii r, as shown schematically in the figure. As a result, the light rays exiting the disk, converge toward the optical axis, and, eventually, cross it in a certain point at a distance f from the disk.



Since the ray bending is relatively small, one may assume that: (i) all the rays travel approximately the same distance b inside the disk; (ii) any given ray enters and exits the disk at approximately the same height r; (iii) $f \gg r$. Thus, the optical pathway $s(r)$ of a ray, incident at a height r above the optical axis, is:

$$s(r) \approx n(r)b + \sqrt{f^2 + r^2} \approx n(r)b + f + \frac{r^2}{2f} \qquad (10)$$

According to Fermat's principle, in order that all the rays focus at the same point, it is necessary that $s(r)$ is constant within $0 \le r < \sigma$. In particular, $s(r) \equiv s(0)$ for any r, which leads to the condition:

$$b(n(0) - n(r)) \equiv \frac{r^2}{2f} \qquad (11)$$

Since:

$$n(0) - n(r) = \gamma(T(0) - T(r)) = -\gamma m r^2 = \gamma |m| r^2 \qquad (12)$$

condition (11) is satisfied if

$$\gamma b |m| r^2 \equiv \frac{r^2}{2f} \qquad (13)$$

The beam hence focuses at

$$f = \frac{1}{2\gamma b |m|} \qquad (14)$$

Taking into account the expression for m derived in part (b), we represent the answer in terms of the known parameters and calculate its numerical value:

$$f = \frac{2\pi k \sigma^2}{\gamma A P} \approx 0.94\,\text{m} \qquad (15)$$

Alternatively, the student may state that the optical pathway $s(r)$ does not depend on r and use the condition $ds/dr \equiv 0$, which gives:

$$\frac{dn(r)}{dr} b + \frac{r}{f} \equiv 0 \qquad (16)$$

Since:

$$\frac{dn(r)}{dr} = \frac{dn}{dT}\frac{dT}{dr} = \gamma 2 m r \qquad (17)$$

we obtain:

$$\frac{r}{f} + 2b\gamma m r \equiv 0 \qquad (18)$$

The beam thus focuses at:

$$f = \frac{1}{2\gamma b |m|} = \frac{2\pi k \sigma^2}{\gamma A P} \approx 0.94\,\text{m} \qquad (19)$$

Task	Pts.
Formulates assumptions (i) and (ii) or equivalent statements	0.2
Derives a general expression for the optical pathway s as a function of r	0.8
Uses that $f \gg r$ and derives approximate quadratic expression for $s(r)$	0.5
States that focusing takes place when $s(r)$ is the same for all rays converging in the focus	0.5
Writes explicitly equation in the form $s(r) \equiv s(0)$ OR $ds/dr \equiv 0$	0.5
Uses the $T(r)$ dependence to derive the $n(r)$ dependence OR to find dn/dr	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**

Approach based on a direct ray/wavefront tracing

As a next approximation, the light ray inside the disk can be modeled as a circular arc of a radius R ($R \gg b$, see the figure). As a result, the ray exits the disk at a smaller height $r - h$ above the optical axis ($h \ll r$). The angle of bending φ of the ray inside the material is related to h by:

$$\cos\varphi = 1 - \frac{h}{R} \qquad (20)$$

From the Snell's law it follows that:

$$n(r)\sin(\pi/2) = n(r-h)\sin(\pi/2-\varphi) = n(r-h)\cos\varphi \quad (21)$$

Up to terms, linear in h, one may write that:

$$\cos\varphi = \frac{n(r)}{n(r-h)} \approx 1 + \frac{n'(r)}{n(r)}h \quad (22)$$

Thus, the radius of the ray inside the material, is:

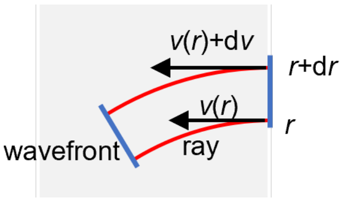
$$R = -\frac{n(r)}{n'(r)} \quad (23)$$

and the bending angle is approximately:

$$\varphi = \frac{b}{R} = -\frac{n'(r)b}{n(r)} \quad (24)$$

Alternatively, the students may trace a small part of the wavefront, associated with two rays, incident at close distances r and $r + \mathrm{d}r$ from the optical axis. By noticing that the wavefront is perpendicular to the rays, the angle φ of deviation of the rays is equal to the angle of rotation of the wavefront. Let $v(r) = c/n(r)$ be the speed of light in the material at a distance r from the axis. The time-rate $\dot\varphi$, i.e. the angular speed of the wavefront is:

$$\dot\varphi = \mathrm{d}v(r)/\mathrm{d}r = -cn'(r)/n(r)^2 \quad (25)$$



The rays reach the opposite surface of the disk in approximately the same time $t = bn(r)/c$, so the total deflection angle of the wavefront, and of the rays, thereof, is $\varphi = \dot\varphi t = -n'(r)b/n(r)$

Upon exiting the disk, the ray undergoes additional refraction, and inclines at a new angle θ relative to the optical axis. The Snell's law in the small-angle approximation ($\sin\theta \approx \theta$, $\sin\varphi \approx \varphi$) states that:

$$\theta = n(r-h)\varphi \approx n(r)\varphi = -n'(r)b \quad (26)$$

Since $n'(r) = \gamma T'(r) = 2mr$, one obtains the following expression for the angle of inclination:

$$\theta = 2\gamma|m|r \quad (27)$$

It is clear from the figure that:

$$f = \frac{r-h}{\tan\theta} \approx \frac{r}{\theta} = \frac{1}{2\gamma|m|b} \quad (28)$$

which reproduces the result obtained by the Fermat's principle.

Task	**Pts.**
States or shows on a graph that the light ray inside the material can be approximated by an arc OR illustrates on a graph that the wavefront is perpendicular to the light rays	0.2
Derives the relation $\cos\varphi = 1 - h/R$ or equivalent OR expresses the time of travel of the ray across the disk	0.5
Applies the Snell's law to the ray path inside the material OR derives a formula for the time-rate $\dot\varphi$ by considering the wavefront passing through two closely separated rays	0.8
Finds expression for the bending angle φ inside the material	0.5
Applies the Snell's law for the refraction of rays exiting the disk	0.5
Uses the $T(r)$ dependence to find $n'(r)$	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**

Alternatively:
Consider a ray incident at a specific height r_0. Let $x \in [0, b]$ be the horizontal coordinate of the ray inside the material. Our goal is to find approximately the profile $r(x)$ of the light ray inside the material. From Snell's law:

$$n(r)\cos\varphi = n(r_0)$$

Since:

$$\tan\varphi = -\frac{\mathrm{d}r(x)}{\mathrm{d}x} = -\frac{\sqrt{(1-\cos^2\varphi)}}{\cos\varphi}$$

we obtain:

$$-\frac{\mathrm{d}r(x)}{\mathrm{d}x} = \frac{\sqrt{(n(r)-n(r_0))(n(r)+n(r_0))}}{n(r)}$$

Since $n(r) \approx n(r_0) + n'(r_0)(r - r_0) = n(r_0) + 2\gamma|m|r_0(r_0 - r)$, the differential equation approximates to:

$$-\frac{\mathrm{d}r(x)}{\mathrm{d}x} = \sqrt{\frac{4\gamma|m|r_0}{n(r_0)}}\sqrt{r0 - r}$$

which is solvable by separation of variables and gives an approximate parabolic path:

$$r(x) = r_0 - \frac{\gamma|m|r_0}{n(r_0)}x^2$$

Finally, for the angle, just before exiting the disk, we obtain:

$$\varphi \approx \tan\varphi = -r'(x=b) = \frac{2\gamma|m|r_0 b}{n(r_0)}$$

Task	**Pts.**
Applies the Snell's law to express the angle φ dependence on r for one ray	0.8
Expresses $\tan\varphi = dr/dx$	0.2
Applies approximation to derive a separable differential equation for $r(x)$	0.8
Integrates to obtain parabolic path	0.5
Finds expression for the angle φ, before exiting the disk	0.2
Applies the Snell's law for the refraction of rays exiting the disk	0.5
Derives an expression for the focal length f in terms of m or of the parameters given in the problem statement	0.8
Calculates f numerically	0.2
Totally on (c)	**4.0**

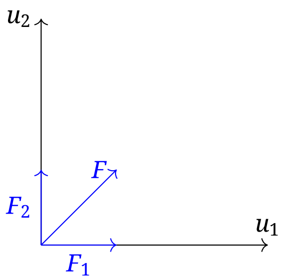
T2: A brick between planes - Solution

(a)

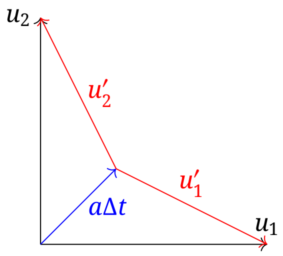
The brick is squeezed between the plates and constrained between the plates, so the normal force on the top and bottom surfaces of the brick must be equal.

Since the brick is moving relative to each plate, the kinetic friction vector on each surface must also be equal in magnitude, but in a direction given by the relative velocity of each plate with respect to the brick. Note that this statement is true whether the plates are moving at the same speed, or different speeds.

A simple vector diagram can illustrate the velocity vectors, the frictional force vectors, and the net force vector.



Then $\vec{a} = \vec{F}/m$, and after a time Δt the new relative velocity vectors will be given by



By symmetry, this continues so that the acceleration of the brick is always at 45 degrees, until eventually,

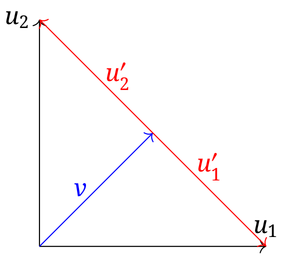


Figure 1: Vector diagram for solution of part (a)

As such, $v = u_1/\sqrt{2}$.

(a)	Pts
Use symmetry to find $u_1' = u_2'$. Must be stated!	1.0
$\vec{u}_1' = -\vec{u}_2'$ at steady state. Must be stated!	1.0
Vector diagram or algebraic equivalent Fig 1	1.0
Correctly find v	1.0
Total on (a)	**4.0**

The two "must be stated" points above need some sort of clear justification, but does not need to be written out in sentences. Just using it in the vector diagram is not sufficient to earn the points.

Special case for students who solve part (b) first

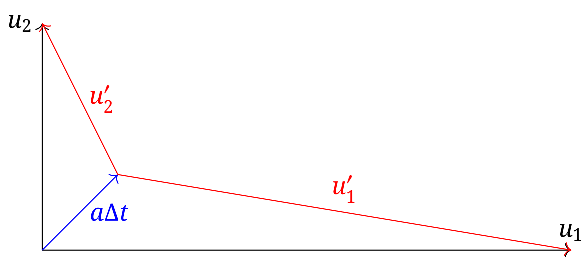
A student who solves part (b) correctly, at least as far as required to solve part (a), and then uses it to write the answer for part (a) correctly will get full marks for part (a). A single trivial mistake in the application of a fully correct part (b) to part (a) will result in half marks for part (a). Trivial mistakes are a clearly identifiable sign error in an an algebra expression, or dropped coefficient between lines. Mistakes that make the answer dimensionally incorrect or physically improbable are not trivial, and would result in 0 marks for part (a).

A student who does not have full marks for part (b), but who used the results of part (b) that has incorrect physics to answer part (a), will get no marks for part (a), even if the answer to part (a) is correct.

(b)

Though the initial vector diagram for velocities looks different, the force components are still equal in magnitude, directed along the relative velocity vectors.

Then $\vec{a} = \vec{F}/m$, and after a time Δt the new relative velocity vectors will be given by



The process is repeated, where the direction of the net force is the angle bisector of the relative velocity vectors $\vec{u}_1'$ and $\vec{u}_2'$

As the quantity $a\Delta t$ is a small quantity compared to the magnitude of the velocity vectors, we can conclude that an equal amount is removed from each of the velocity vectors, so that magnitude comparison

$$u_1 - u_2 = u_1' - u_2' \tag{29}$$

is a conserved quantity.

For the next step, the net force vector is still the angle bisector for u_1 and u_2. The above arguments will hold true for the conserved quantity, and then the final velocity can be found by

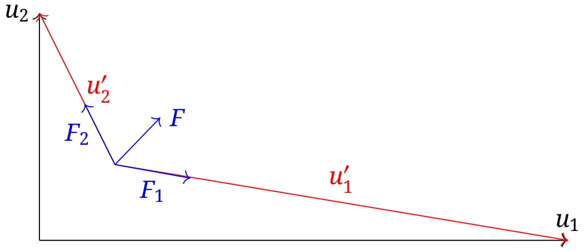


Figure 2: The correct construction to find direction of net force

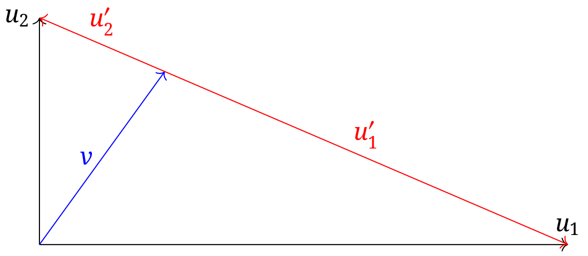


Figure 3: Vector diagram for solution of part (b)

Then

$$(u_1' + u_2')^2 = u_1^2 + u_2^2 = D^2 \tag{30}$$

and

$$u_1' - u_2' = u_1 - u_2 = \delta \tag{31}$$

can be solved for u_1' and u_2', with

$$u_2' = \frac{1}{2}(D - \delta)$$

and

$$u_1' = \frac{1}{2}(D + \delta)$$

This gives the components of v as

$$v_x = \frac{u_2'}{D}u_1 = \frac{u_1}{2}\left(1 - \frac{\delta}{D}\right) \tag{32}$$

and

$$v_y = \frac{u_1'}{D}u_2 = \frac{u_2}{2}\left(1 + \frac{\delta}{D}\right) \tag{33}$$

and the magnitude is

$$v = \frac{1}{2}\sqrt{D^2 + \delta^2 - 2\frac{\delta}{D}\left(u_1^2 - u_2^2\right)} \tag{34}$$

Writing this in terms of u_1 and u_2 is left as an exercise for the reader.

$$v = \sqrt{\frac{1}{2}\left(u_1^2 + u_2^2 - u_1u_2 - \frac{(u_1 - u_2)^2(u_1 + u_2)}{\sqrt{u_1^2 + u_2^2}}\right)}$$

$$= \frac{1}{\sqrt{2}}\sqrt{u_1u_2 - (u_1 - u_2)^2\left(\frac{u_1 + u_2}{\sqrt{u_1^2 + u_2^2}} - 1\right)}$$

(b) Scheme G1	**Pts**
Show/explain that $\vec{a}$ is always angle bisector	1.0
Show that δ (Eq 29) is a constant of the motion	1.0
$\vec{u}_1'$ and $\vec{u}_2'$ are in opposite directions at steady state	1.0
Vector diagram for relative velocities Fig 3	1.0
Apply D (Eq 30) or equivalent	0.2
Apply δ (Eq 31) or equivalent	0.2
Correctly find v_x or v_y	0.4
Correctly find other one	0.2
Correctly find v	1.0
Total on (b)	**6.0**

A student who correctly finds v without explicitly writing v_x and v_y will be assumed to have found v_x and v_y by applying D and δ and should get those points.

An incorrect value for v_x or v_y or v that is dimensionally correct and wrong only because of a math error will get half marks, but the math error must be clear.

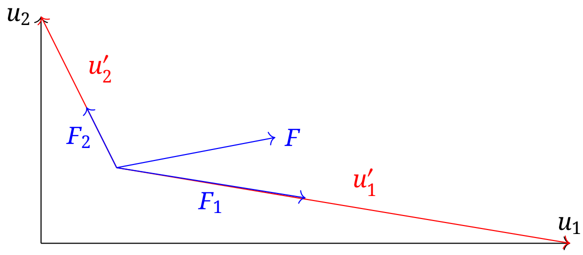
Writing v in an equivalent form of Eq. 34 but not in the final form in terms of u_1 and u_2 only receives only 0.6/1.0 points for last part.

Possible Error Scenarios

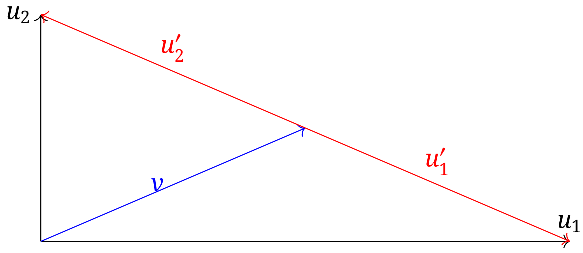
Students who attempt graphical approach who have an error below should follow the grading scheme below.

1. **Assuming that the direction of the frictional force is given by the vector sum of the relative velocities.**

   An example of the improper vector construction is shown in the following figure.



   The immediate consequence is that Eq. 29 is no longer true. With quick inspection, the student should conclude that the final state must have $\vec{u}_1 = -\vec{u}_2$, and then the graphical velocity picture would be, at steady state,



(b) Scheme E1	Pts
Vector diagram showing (incorrect!) force direction, or equivalent statement	0.5
A clear statement that the (incorrect) force direction depends on the vector sum of the relative velocities (see note below!)	0.5
Explicitly stating (incorrect!) $\vec{u}_1' = -\vec{u}_2'$ at steady state.	0.5
Vector diagram for steady state, or equivalent statement	0.8
Stating $v = \frac{1}{2}\sqrt{u_1^2 + u_2^2}$	0.7
Total on (b)	**3.0**

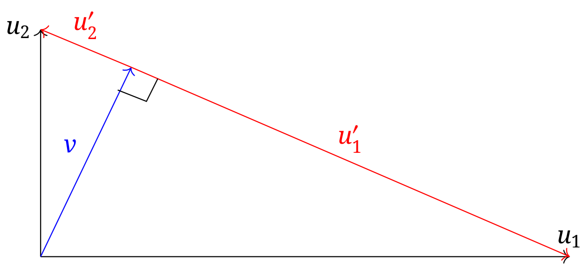
There is no other partial credit possible for this approach.

Note: the student must demonstrate knowledge that the force is proportional to the relative velocity; if they only look at the force at the start of the problem, and never address what happens as the problem evolves, they don't get this 0.5 points.

2. **Assuming that the direction of the final velocity is orthogonal to the vector difference of the velocities of the plates.**

This scheme only applies when a student explicitly states that v is perpendicular. Do not use this grading scheme if it is simply a vaguely drawn vector diagram.

It is true that the acceleration vector approaches the steady state orthogonal to the hypotenuse of the triangle, but not the final velocity. As such, the following vector diagram is wrong:



It is possible that the student started with some correct physics according to the original approach, and then makes the above assumption to finish the problem; it is also possible that they started with the perpendicular assumption.

If the only mistake is a bad final vector diagram with a perpendicular marking, but then did the algebra based on D and δ then use the following:

(b) Scheme E2A	Pts
Show/explain that $\vec{a}$ is always angle bisector	1.0
Show that δ (Eq 29) is a constant of the motion	1.0
$\vec{u}_1'$ and $\vec{u}_2'$ are in opposite directions at steady state	1.0
Vector diagram above (incorrect!) for relative velocities at steady state, or any statement that final v is orthogonal.	0.5
Apply D (Eq 30) or equivalent Apply δ (Eq 31) or equivalent Correctly find v_x Correctly find v_y	0.2 0.2 0.3 0.3
Correctly find v	1.0
Total on (b)	**5.5**

A student who correctly finds v without explicitly writing v_x and v_y will be assumed to have found v_x and v_y by applying D and δ and should get those points.

An incorrect value for v_x or v_y or v that is dimensionally correct and wrong only because of a math error will get half marks, but the math error must be clear.

Writing v in an equivalent form of Eq. 34 but not in the final form in terms of u_1 and u_2 only receives only 0.6/1.0 points for last part.

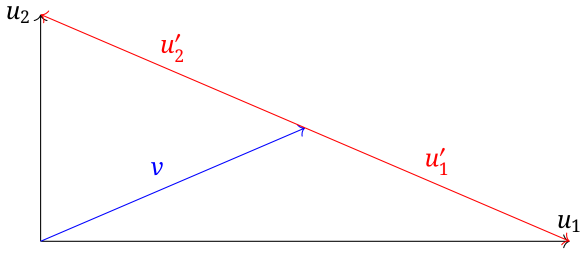
If the mistake is that they used the perpendicular vector diagram to solve the problem, then their answers would be different, so use the following:

(b) Scheme E2B	Pts
Show/explain that $\vec{a}$ is always angle bisector	1.0
Show that δ (Eq 29) is a constant of the motion	1.0
$\vec{u}_1'$ and $\vec{u}_2'$ are in opposite directions at steady state	1.0
Vector diagram above (incorrect!) for relative velocities at steady state, or any statement that final v is orthogonal.	0.5
Find (incorrect!) $v = u_1 u_2 / \sqrt{u_1^2 + u_2^2}$	1.0
Total on (b)	**4.5**

A dimensionally correct formula for v that differs from above because of a single math error in a clear derivation based on the figure will get 0.5 out of 1.0 for the v.

3. **Assuming that the final velocity is the average of the vector velocities of the plates.**

Note that this approach yields the same result as a previous approach, but it is not worth as many points, as the fundamental physics starts from a higher level incorrect assumption. Here, the student is just assuming that the final speed is an average, in the previous approach the student had an error in the direction of the forces. If they use forces, a previous grading scheme applies.

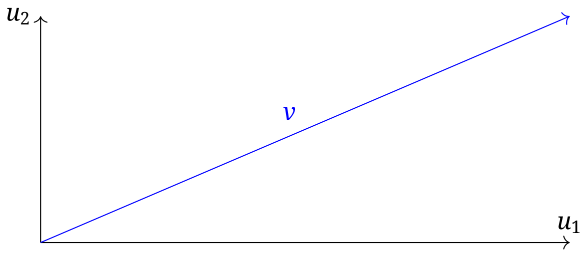


(b) Scheme E3	Pts
Explicity stating $\vec{u}_1'$ and $\vec{u}_2'$ are in opposite directions at steady state.	1.0
Vector diagram above	0.5
Stating $v = \frac{1}{2}\sqrt{u_1^2 + u_2^2}$	0.5
Total on (b)	**2.0**

There is no other partial credit possible for this approach.

4. **Assuming that the final velocity is the vector sum of the velocities of the plates.**

   Though very tempting, it violates so many requirements for a steady state solution that this is not worth very many points at all.



(b) Scheme E4	Pts
Vector diagram above	0.5
Total on (b)	**0.5**

There are no points for finding an expression for v, as the method is just wrong.

Alternative Approaches

Non-Cartesian Differential Equations Effectively a graphical approach without graphics, one can focus on the relative velocity vectors as coordinate axis. Then the important equations of motion are

$$\frac{du_{1r}}{dt} = -\frac{F}{m} \text{ and } \frac{du_{2r}}{dt} = -\frac{F}{m} \tag{35}$$

where the force magnitude F is a function of the relative directions of the coordinate axes $\vec{u}_{1r}$ and $\vec{u}_{2r}$.

The student can quickly realize that the difference of these two expressions is zero, so that δ is a constant of the motion.

At steady state, $F = 0$, and this happens when $\vec{u}_{1r}$ and $\vec{u}_{2r}$ point in opposite directions.

(b) Scheme A1	Pts
A set of differential equations Eq35	1.0
Show that δ (Eq 29) is a constant of the motion	1.0
$\vec{u}_1'$ and $\vec{u}_2'$ are in opposite directions at steady state	1.0
Equivalent math statement or diagram for steady state relative velocities Fig 3	1.0
Apply D (Eq 30) or equivalent	0.2
Apply δ (Eq 31) or equivalent	0.2
Correctly find v_x or v_y	0.4
Correctly find other one	0.2
Correctly find v	1.0
Total on (b)	**6.0**

A student who correctly finds v without explicitly writing v_x and v_y will be assumed to have found v_x and v_y by applying D and δ and should get those points.

An incorrect value for v_x or v_y or v that is dimensionally correct and wrong only because of a math error will get half marks, but the math error must be clear.

Writing v in an equivalent form of Eq. 34 but not in the final form in terms of u_1 and u_2 only receives only 0.6/1.0 points for last part.

Follow on errors are not allowed for first parts, as the math expressions are almost trivial. They are also not allowed for the algebraic part at the end; see possible mistakes above for possible scenarios where points could be awarded by writing an equivalent algebraic expression for the graphical approach.

Cartesian Differential Equations Attempting to set up equations of motions in a Cartesian system requires finding the direction of relative velocity of each surface. Assuming that u_1 is in the x direction and u_2 is in the y direction, and if the velocity components of the block are v_x and v_y, the relative velocities of the planes are

$$u_{1rx} = u_1 - v_x \text{ and } u_{1ry} = -v_y$$

and

$$u_{2rx} = -v_x \text{ and } u_{2ry} = u_2 - v_y$$

The forces of friction from each plane are equal in magnitude and directed along the relative velocity vectors, so a steady state solution is when these two relative vectors are in opposite directions.

The force vectors then have components

$$F_{1x} = F\frac{u_{1rx}}{u_{1r}} \text{ and } F_{1y} = F\frac{u_{1ry}}{u_{1r}}$$

and

$$F_{2x} = F\frac{u_{2rx}}{u_{2r}} \text{ and } F_{2y} = F\frac{u_{2ry}}{u_{2r}}$$

This gives the following equation of motion for the block:

$$\frac{dv_x}{dt} = a_x = \frac{F}{m}\left(\frac{u_{1rx}}{u_{1r}} + \frac{u_{2rx}}{u_{2r}}\right)$$

and

$$\frac{dv_y}{dt} = a_y = \frac{F}{m}\left(\frac{u_{1ry}}{u_{1r}} + \frac{u_{2ry}}{u_{2r}}\right)$$

A student might have noticed that this is a nasty set of coupled differential equations.

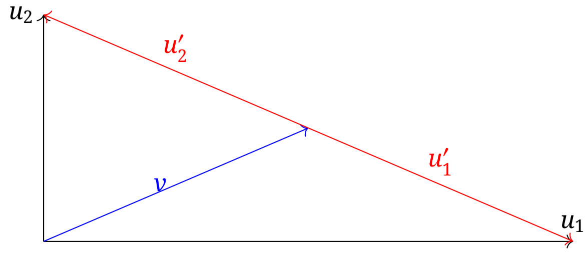
Both of these have vanishing accelerations according to the same condition:

$$u_{1rx}^2 u_{2ry}^2 = u_{2rx}^2 u_{1ry}^2$$

which is merely the statement that at steady state the two relative velocity vectors are in opposite directions. As this is still only one equation, it is not possible to find the steady state v_x and v_y from this alone. A simpler expression can be found, however,

$$u_1 u_2 = u_1 v_y + u_2 v_x. \tag{36}$$

This, however, is merely the statement of the graphical vector diagram



Still, it is not possible to know where the vector $\vec{v}$ touches the line described by $\vec{u}_1 - \vec{u}_2$.

This isn't the end, however. Consider

$$u_{1r}^2 = u_{1rx}^2 + u_{1ry}^2$$

Taking the time derivative, one gets

$$u_{1r}\dot{u}_{1r} = u_{1rx}\dot{u}_{1rx} + u_{1ry}\dot{u}_{1ry}$$

Combine with the above, and

$$u_{1r}\dot{u}_{1r} = -\frac{F}{m}\left(\frac{u_{1rx}^2}{u_{1r}} + \frac{u_{1rx}u_{2rx}}{u_{2r}} + \frac{u_{1ry}^2}{u_{1r}} + \frac{u_{1ry}u_{2ry}}{u_{2r}}\right)$$

which means

$$\dot{u}_{1r} = -1 - \frac{u_{1rx}u_{2rx}}{u_{1r}u_{2r}} - \frac{u_{1ry}u_{2ry}}{u_{1r}u_{2r}}$$

Out of symmetry, an identical expression will be found for $\dot{u}_{2r}$, which means that

$$\delta = u_{1r} - u_{2r} \tag{37}$$

is a constant of the motion.

At this point, a student can apply the results of Eq 36 with this constant difference formula and solve for v_x and v_y as done in the first solution.

In the scheme below, finding the expression assumes found correctly and completely, and a student would get 0.2 points for (most) expressions. There are no partial points if the equation is wrong.

However, for the first four categories only (marked with an *, follow on errors are not penalized for work based on a previous mistake, assuming that it does not trivialize the result.

(b) Scheme A2	**Pts**
Expressions for relative velocity components (4 @ 0.2 each, *)	0.8
Expressions for force components (4 @ 0.2 each, *)	0.8
Expressions for acceleration components (2 @ 0.2 each, *)	0.4
State a condition for steady state Find equation for steady state or equivalent to Eq 36 (must not have relative velocity, or is incomplete, *)	0.5 0.5
Show that δ (Eq 37) is a constant of the motion (must be correct, regardless of follow on error)	1.0
Apply D (Eq 30) or equivalent Apply δ (Eq 31) or equivalent Correctly find v_x or v_y Correctly find other one	0.2 0.2 0.4 0.2
Correctly find v	1.0
Total on (b)	**6.0**

A student who correctly finds v without explicitly writing v_x and v_y will be assumed to have found v_x and v_y by applying D and δ and should get those points.

An incorrect value for v_x or v_y or v that is dimensionally correct and wrong only because of a math error will get half marks, but the math error must be clear.

Writing v in an equivalent form of Eq. 34 but not in the final form in terms of u_1 and u_2 only receives only 0.6/1.0 points for last part.

T3: Plate between magnets - Solution

General remarks:

Calculational numerical errors - 0.2 p Dimensionally wrong answers - 0 p

Part a) (~3 pts)

Let us analyze how the current in the metal starts to build up when we start moving the plate in the rest frame of the magnets. The free electrons in the metal try to move together with the plate in the $+y$ direction, but a Lorentz force pointing in the $-x$ direction makes them deviate.

After some time the charge accumulation stops and the charge and current distributions do not depend on time anymore. According to Ohm's law,

$$\vec{j} = \vec{j}_P + \frac{1}{\varrho}\vec{v} \times \vec{B},$$

where $\vec{j}$ is the total current density, $\vec{v} \times \vec{B}$ is the result of the Lorentz force, and $\vec{j}_P$ is the current density caused by the electric field (and is therefore a potential field).

Inside the circular region ($r < R$) the Lorentz term is constant, but outside it is zero. Nevertheless, the E-field of the accumulated charges is present there, which gives rise to a current. In the steady state considered here, the charge distribution is constant in time, and hence the total current density $\vec{j}$ is source-free everywhere. However, this does not hold for $\vec{j}_P$ which is driven by the "sources" and "sinks" along the rim of the circular region of radius R. Those sources and sinks compensate exactly the sources of the Lorentz term mentioned above. Only a source-free jump of the total current density is allowed on the rim, resulting in the kinks on the streamlines along the boundary.

Based on these considerations, we can draw a qualitative pattern of the current flow: inside the circular disc, there is a flow in the direction of the Lorentz force and the streamlines of the created flow of current close onto themselves outside of that region as shown in the figure below.

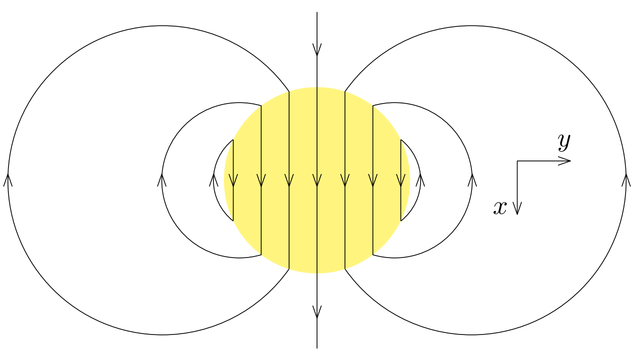


fig. 1

Grading scheme part a)	Pts
Streamlines are continuous **and** arrows are consistent along each streamline **and** streamlines don't meet or intersect. If the streamlines are qualitatively (physically) wrong, 0 p should be given.	1.0
Twofold symmetry* with respect to x-axis (if axis is not indicated in any way, only 0.1 p)	0.2
Twofold symmetry with respect to y-axis (if axis is not indicated in any way, only 0.1 p)	0.2
Indicating the boundary of the circular region	0.1
Breaking points on the boundary (only rewarded if both inside and outside streamlines are shown)	0.8
Along the y-axis the direction of current density is $+x$ inside and $-x$ outside (if axis is not indicated in any way, 0.0 p; if the direction is correct in only one rigion, 0.3 p; if there is a sign mistake only, 0.3 p)	0.7
Stating that current density is homogeneous inside is not required in part a)	0.0
Total on part a)	**3.0**

Part b (~5 pts)

From here on, we make use of the uniqueness of the solution to the correctly posed boundary conditions for the Maxwell equations inside the plate. We need to satisfy the condition that current density is sourceless everywhere, i.e.

$$\oint \vec{j} \times \vec{dl} = 0, \tag{38}$$

where $\vec{j}$ denotes the current density and integral is taken over an arbitrary closed loop inside the plate. The current density must satisfy the Ohm's law in differential form,

$$\vec{j} = \rho^{-1}(\vec{E} + \vec{v} \times \vec{B}), \tag{39}$$

where $\vec{v}$ denotes the velocity of the plate; the electric field caused by the accumulated charges $\vec{E}$ must satisfy the circulation theorem,

$$\oint \vec{E} \cdot \vec{dl} = 0, \tag{40}$$

where integration loop can be arbitrary. To close the system of equations we need to satisfy, we integrate Eq. 39 either over a closed loop entirely inside or entirely outside the circular region then we obtain zero, because for such loops $\oint \vec{v} \times \vec{B} \cdot \vec{dl} = 0$. From this we conclude that for such integration loops,

$$\oint \vec{j} \cdot \vec{dl} = 0. \tag{41}$$

Note that the electric field $\vec{E}_{\rm pol}$ of a polarised cylinder satisfies Eq. 40, but we must also satisfy

Eqns. (38,39). We will look for such a solution in the form of a superposition

$$\vec{j} = \rho^{-1}(\vec{E}_{\text{pol}} + \vec{v} \times \vec{B}). \quad (42)$$

First we need to find an expression for the field $\vec{E}_{\text{pol}}$. To that end, we can consider the electric field produced by an infinite homogeneously charged cylinder of radius R and volume charge density q. From Gauss's law, we can conclude that inside and outside the cylinder the electric field is

$$\vec{E}_+(\vec{r}) = \begin{cases} \frac{1}{2\varepsilon_0} q\vec{r} & \text{if } r \le R, \\ \frac{1}{2\varepsilon_0} q \frac{R^2}{r^2}\vec{r} & \text{if } r > R. \end{cases}$$

Now we can take another similar cylinder of charge density $-q$, displaced from the first cylinder by a small displacement vector $\vec{s}$. With the origin still at the centre of the positively charged cylinder, the electric field of this negatively charged cylinder is written as $-\vec{E}_+(\vec{r}-\vec{s})$ (which works for both inside and outside). Note that in the overlapping area of the two cylinders, the charge density is zero.

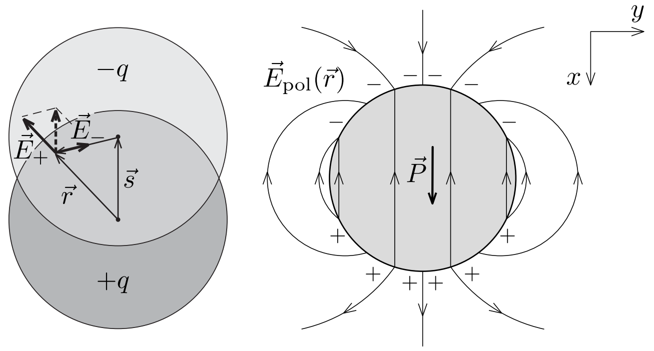


fig. 2

We consider the limit $|\vec{s}| \to 0$ while keeping the product $q\vec{s} = -\vec{P}$ constant, which represents the volume density of the dipole moments (i.e. the polarisation), constant. In this case, the field inside can be calculated as

$$\vec{E}_{\text{inside}} = \vec{E}_+(\vec{r}) + \vec{E}_-(\vec{r}) = \vec{E}_+(\vec{r}) - \vec{E}_+(\vec{r}-\vec{s}) = -\frac{1}{2\varepsilon_0}\vec{P}, \quad (43)$$

which is homogeneous.

Outside, the field is given by

$$\vec{E}_{\text{outside}} = \vec{E}_+(\vec{r}) - \vec{E}_+(\vec{r}-\vec{s}) = \frac{1}{2\varepsilon_0} qR^2 \left(\frac{\vec{r}}{r^2} - \frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^2} \right).$$

Expanding the second term in the bracket up to first order ($|\vec{s}| \ll |\vec{r}|$):

$$\frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^2} \approx \frac{\vec{r}-\vec{s}}{r^2 - 2\vec{r}\vec{s}} \approx \frac{\vec{r}-\vec{s}}{r^2}\left(1 + 2\frac{\vec{r}\vec{s}}{r^2}\right).$$

Substituting this into the formula of $\vec{E}_{\text{outside}}$ and simplifying:

$$\vec{E}_{\text{outside}} = \frac{1}{2\varepsilon_0} R^2 \frac{2\vec{r}(\vec{r}\vec{P}) - r^2\vec{P}}{r^4}.$$

This is the field of an ideal two-dimensional electric dipole. From this general formula we can obtain an expression for the field along the y axis:

$$\vec{E}_{\text{outside}} = -\frac{1}{2\varepsilon_0}\vec{P}\left(\frac{R}{r}\right)^2. \quad (44)$$

The normal component of this field exhibits discontinuity σ/ε at the boundary of the cylinder given by the surface charge density σ, which can be found again from the superposition of the two cylinders: the non-overlapping region has the shape of a crescent moon of width s in the x direction; this means that the width in the radial direction is $t = s\cos\varphi$, where φ denotes the angle between the radius vector and the $x-axis$. Hence, we can find the surface charge density as $\sigma = qt = P\cos\varphi$. Due to Gauss' law, the radial electric field jumps by an amount given by the surface charge density, $\{E_r\} = \sigma/\varepsilon_0$. This corresponds to a jump in the magnitude of the current density

$$\{j\} = \rho^{-1}\sigma/\varepsilon_0 = \rho^{-1}\sigma P\cos\varphi \quad (45)$$

which must eliminate the discontinuity due to the jump in the radial component of the Lorentz force term in Eq. (42, equal to

$$\{\rho^{-1}(v \times B)_r\} = -vB\cos\varphi. \quad (46)$$

The cancellation occurs when $\vec{P} = -\vec{e}_x vB\varepsilon_0$. Putting all the results together, we can conclude that inside the cylinder, the current density in the plate is homogeneous and equal to

$$\vec{j} = \frac{1}{2}\frac{\vec{v} \times \vec{B}}{\varrho}. \quad (47)$$

Outside the cylinder, along the y axis, $\vec{B} = 0$ so that

$$\vec{j} = \frac{\vec{E}_{\text{outside}}}{\varrho} = -\frac{1}{2\varepsilon_0}\vec{P}\left(\frac{R}{r}\right)^2 = -\frac{1}{2}\frac{\vec{v} \times \vec{B}}{\varrho}\left(\frac{R}{r}\right)^2. \quad (48)$$

Now we can plot the graph:

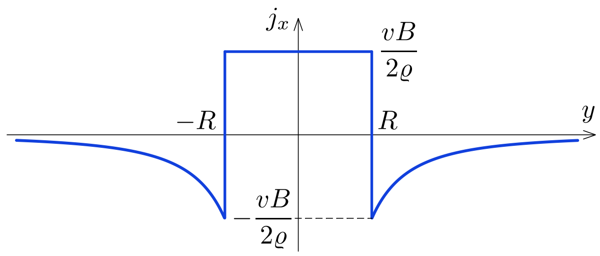


fig. 3

Grading scheme part b)	Pts
Realizing that $\vec{j}$ is driven by the sum of the term $(\vec{v}\times\vec{B})/\varrho$ **and** the electric field (if only one term is considered, 0 p)	0.5
Writing the boundary condition for $\vec{j}_P$ and for $\vec{j}$ (or equivalent)	0.5
Guessing the solution of the 2D boundary value problem (either with analogy of a polarized dielectric cylinder or with the superposition of a 2D dipole field outside and a uniform field inside)	1.0
Calculating the electric field of a 2D dipole along the symmetry axis	1.0
Finding the appropriate weight of the dipole field (i.e. dipole moment) by matching boundary conditions	0.5
Expressing $\vec{j}$ inside and outside correctly	0.5
Plotting the graph (axis are labelled correctly, 0.1+0.1; sign changes at the rim and function jumps, 0.3, function is constant in region $[-R,R]$ 0.2, if the area below the curve could add up to zero 0.3)	1.0
Total on part b)	**5.0**

Part c(~2 pts)

Method 1: integrating the force density. Only the current inside the magnetic field contributes to the force. The force acting on an infinitesimal volume element $\mathrm{d}V$ of the metal slab is given by

$$\mathrm{d}\vec{F} = \vec{j}_{\text{inside}} \times \vec{B}\mathrm{d}V\,.$$

Since $\vec{j}_{\text{inside}}$ and $\vec{B}$ are uniform in the region of non zero B, and perpendicular, the integral simplifies to

$$\vec{F} = \vec{j}_{\text{inside}}\vec{B}\int \mathrm{d}V = \vec{j}_{\text{inside}} \times \vec{B}\,V = \vec{j}_{\text{inside}} \times \vec{B}\,\pi R^2\delta\,.$$

Using our previous result for the current density we get

$$\vec{F} = \frac{\pi R^2\delta}{2\varrho}(\vec{v}\times\vec{B})\times\vec{B} = -\frac{\pi R^2\delta}{2\varrho}B^2\vec{v}\,.$$

The direction of the force is opposite to the velocity of the plate, in agreement with physical intuition.

Grading of part c) - Method 1	Pts
Expressing the force acting on a current element	0.5
integrating over the circular region	1.0
final result (it can be given only if prefactor is correct)	0.5
Total on part c)	**2.0**

Method 2: integrating the dissipated power.

Due to energy conservation, the mechanical power provided by the external force is dissipated in the plate in the form of Joule heat:

$$Fv = P_{\text{dissipated}}\,.$$

The volume density of the dissipated power is ϱj^2, so the total power of dissipation can be written as an integral over the volume:

$$P_{\text{dissipated}} = \int_{\text{in}} \varrho j^2_{\text{inside}}\,\mathrm{d}V + \int_{\text{out}} \varrho j^2_{\text{outside}}\,\mathrm{d}V\,.$$

The first integral simplifies due to the constant current density inside the circular region of radius R:

$$\int_{\text{in}} \varrho j^2_{\text{inside}}\,\mathrm{d}V = \varrho j^2_{\text{inside}}V = \frac{1}{4\varrho}v^2B^2\cdot\pi R^2\delta\,.$$

In order to evaluate the second integral, let us calculate j^2_{outside} first:

$$j^2_{\text{outside}} = \frac{1}{4\varrho^2}R^4\frac{|r^2\vec{v} - 2\vec{r}(\vec{r}\vec{v})|^2}{r^8}B^2\,,$$

where the absolute value squared can be simplified as

$$|r^2\vec{v} - 2\vec{r}(\vec{r}\vec{v})|^2 = r^4v^2 - 4r^2(\vec{r}\vec{v})(\vec{r}\vec{v}) + 4r^2(\vec{r}\vec{v})(\vec{r}\vec{v}) = r^4v^2$$

so the quantity j^2_{outside} depends on the distance r only (and not on the polar angle):

$$j^2_{\text{outside}} = \frac{1}{4\varrho^2}R^4\frac{v^2)}{r^4}B^2\,.$$

Now the power dissipated outside the circular region $(r > R)$ can be written as:

$$\int_{\text{out}} \varrho j^2_{\text{outside}}\,\mathrm{d}V = \int_R^\infty \frac{1}{4\varrho^2}R^4\frac{v^2}{r^4}B^2\cdot 2\delta\pi r\,\mathrm{d}r$$

Let us evaluate the integral:

$$\int_{\text{out}} \varrho j^2_{\text{outside}}\,\mathrm{d}V = \frac{v^2B^2\pi R^4\delta}{2\varrho}\int_R^\infty \frac{\mathrm{d}r}{r^3} = \frac{v^2B^2\pi R^4\delta}{2\varrho}\frac{1}{2R^2}\,.$$

It is an interesting coincidence that the power dissipated inside and outside is the same. Now the total power of dissipation is known and the force can be calculated:

$$F = \frac{P_{\text{dissipated}}}{v} = \frac{\pi R^2\delta}{2\varrho}B^2v\,.$$

Grading of part c) - Method 2	Pts
Expressing the force in terms of the dissipated power	0.5
integrating over the circular region	0.3
integrating over the outer region	0.7
final result (it can be given only if prefactor is correct)	0.5
Total on part c)	**2.0**

An incorrect method of obtaining the correct answer. The correct solution described above can be carried out with minor modifications when using the reference frame of the plate, where the circular region of the magnetic field moves with velocity $-\vec{v}$.

The only difference is that the Lorentz force $e\vec{v}\times\vec{B}$ acting on the charge carriers in the cylinder frame is now replaced by the Coulomb force $e\vec{E}_B$, where $\vec{E}_B = \vec{v}\times\vec{B}$ is the electric field resulting now from the Lorentz transformation.

What we need to notice is that there is no electric field outside of the cylinder ($r > R$) at large distances in the z direction from the plate. Indeed, there are clearly no fields there (neither $\vec{B}$ nor $\vec{E}$) in the cylinder's frame. Using the Lorentz transformation it is clear that no fields can emerge in our new frame. Meanwhile, outside the cylinder and inside the plate, there is definitely an electric field driving the current flow as derived above. The tangential-to-the-plate electric field cannot be discontinuous at the surface because the electrostatic field (in the cylinder's frame) is potential. Hence, outside the cylinder and outside the plate, there is also an electric field in the neighbourhood of the plate, It is caused by the charge density formed on and in the plate, the generation of which is described in the solution of Part a). From large distances, however, these charges are seen as an electric dipole, whose field vanishes inversely proportional to the cubed distance.

A tempting but incorrect approach is trying to describe the motion of the cylinder in the plate's frame as a discontinuous stepping motion of the cylinders: at the current position of the cylinder, the magnetic flux disappears and at the same time reappears at a small distance away from its previous position. According to the Faradey's law, the disappearing flux will produce circular clockwise fieldlines of $\vec{E}$, and the reappearing flux will produce slightly displaced counterclockwise fieldlines. The corresponding fields and their superposition $\vec{E}_s(\vec{r})$ can be easily calculated, see below. The result turns out to be exactly the same as the field $\vec{E}_o(\vec{r})$ obtained above for the field inside the plate. However, the equivalence of these fields is just a coincidence (there is a good reason for this coincidence: in both cases, the field is obtained as a superposition of the fields produced by two overlapping cylinders). Furthermore, there is a fundamental difference: in reality, the field $\vec{E}_o(\vec{r})$ is only observed in the plate (at $z \approx 0$) and disappears at $z \gg R$; meanwhile, in the stepping cylinder model, the field $\vec{E}_s(\vec{r})$ remains the same regardless of the value of z.

The reason why the stepping cylinder model gives an incorrect result — producing an electric field in those places where there clearly is none — lies in the theory of relativity. A continuously moving (not stepping) cylinder obtains a surface charge, because the current density and the charge density together form a 4-vector. There is a bound surface current on the surface of the cylinder (creating the magnetic field inside), and as the cylinder moves, the Lorentz transformation tells us that there will now also be a surface charge creating an electric field both inside and outside the cylinder. So, with a moving cylinder, the changing flux creates an electric field outside which is cancelled out by the surface charges on the cylinder. Inside the cylinder, these two fields add up constructively.

In what follows, we analyse mathematically the model of stepping cylinders. By considering two almost overlapping circles with a short time interval τ between, it is clear that any flux change occurs on rim (see fig. 1). This changing flux gives rise to an induced electric field which makes the charge carriers move inside the plate resulting electrical currents.

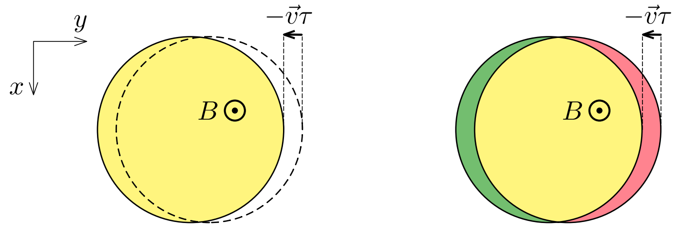


fig. 1

Considering the red region in the figure as the initial non-zero B field, and the green region as the non-zero B field after time τ, the yellow overlapping region corresponds to a non changing magnetic flux, while red on the right is decreasing, and green on the left is increasing. The average rate of change of B-field in the two regions is $\pm B/\tau$, respectively.

The induced electric field can be found as a superposition of the electric fields generated by the flux change in the two overlapping circular regions. For one circular region with increasing B-field in the $+z$ direction, the magnitude of the electric field can be found from Faraday's law applied for a circular E-field line:

$$E_+(r)\cdot 2\pi r = \frac{\mathrm{d}\Phi}{\mathrm{d}t},$$

where the rate of enclosed flux change is given by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \begin{cases} \pi r^2 \dfrac{B}{\tau}, & \text{if } r \le R, \\ \pi R^2 \dfrac{B}{\tau}, & \text{if } r > R. \end{cases}$$

From here the modulus of the induced electric field can be expressed. It is convenient to write the result in the vectorial form with the help of position vector $\vec{r}$, unit vector $\vec{e}_z$ and Lenz's law:

$$\vec{E}_+(\vec{r}) = \begin{cases} \dfrac{1}{2}\dfrac{B}{\tau}\vec{r}\times\vec{e}_z, & \text{if } r \le R, \\ \dfrac{1}{2}\dfrac{B}{\tau}\dfrac{R^2}{r^2}\vec{r}\times\vec{e}_z, & \text{if } r > R. \end{cases}$$

A similar formula gives the E-field generated by the region of decreasing magnetic flux, we only need to change the sign and replace $\vec{r}$ with $\vec{r}-\vec{v}\tau$.

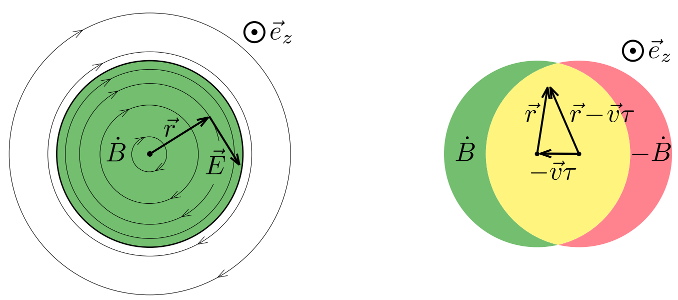


fig. 2

Now take the superposition of the induced electric fields produced by the two overlapping circular regions. Between the two magnets ($r \le R$) we get

$$\vec{E}_{\text{inside}} = \vec{E}_+ + \vec{E}_- = \frac{1}{2}\frac{B}{\tau}\vec{r} \times \vec{e}_z - \frac{1}{2}\frac{B}{\tau}(\vec{r} - \vec{v}\tau) \times \vec{e}_z = \frac{1}{2}\vec{v} \times \vec{e}_z B \,.$$

Using the differential form of Ohm's law $\vec{j} = \vec{E}/\varrho$, we can conclude that the current density in the metal plate in the region $r \le R$ is uniform,

$$\vec{j}_{\text{inside}} = \frac{1}{2\varrho}\vec{v} \times \vec{B} \,,$$

where we used that $\vec{B} = B\vec{e}_z$.

Similarly, in the region $r > R$ we get:

$$\vec{E}_{\text{outside}} = \vec{E}_+ + \vec{E}_- = \frac{1}{2}\frac{B}{\tau}R^2 \left[\frac{\vec{r}}{r^2} - \frac{\vec{r} - \vec{v}\tau}{|\vec{r} - \vec{v}\tau|^2}\right] \times \vec{e}_z .$$

Expanding the second term in the bracket up to first order ($|\vec{v}\tau| \ll |\vec{r}|$):

$$\frac{\vec{r} - \vec{v}\tau}{|\vec{r} - \vec{v}\tau|^2} \approx \frac{\vec{r} - \vec{v}\tau}{r^2 - 2\vec{r}\vec{v}\tau} \approx \frac{\vec{r} - \vec{v}\tau}{r^2}\left(1 + 2\frac{\vec{r}\vec{v}\tau}{r^2}\right) .$$

Substituting this into the formula of $\vec{E}_{\text{outside}}$ and simplifying:

$$\vec{E}_{\text{outside}} = \frac{1}{2}BR^2\frac{r^2\vec{v} - 2\vec{r}(\vec{r}\vec{v})}{r^4} \times \vec{e}_z \,.$$

Finally, the current density in the region $r > R$ is the following

$$\vec{j}_{\text{outside}} = \frac{1}{2\varrho}R^2\frac{r^2\vec{v} - 2\vec{r}(\vec{r}\vec{v})}{r^4} \times \vec{B}$$

Grading of incorrect approach	**Pts**
part a) - same as the original marking scheme	3.0
If everything is done correctly - 2.0 p	2.0
part c) - same as the original marking scheme	2.0
Total for incorrect approach	**7.0**

Note no. 1. Although it was not asked in the problem, it is fascinating that the streamlines in the region $r > R$ are circles. To prove this statement, take a circle of radius a with center located at vector $\vec{a}$ such that $\vec{a}\vec{p} = 0$ (see fig. 3). The position vectors of the points located on this circle satisfy the equation

$$|\vec{r} - \vec{a}|^2 = a^2 \,,$$

Which can be rearranged in the form

$$2\vec{a}\vec{r} - r^2 = 0$$

We will show that $\vec{j}_{\text{outside}}$ is tangent to such a circle, regardless of the radius a.

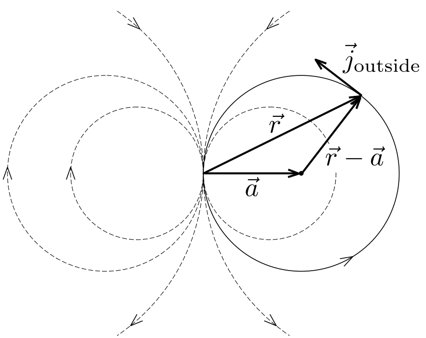


fig. 3

Consider the scalar product

$$\vec{j}_{\text{outside}}(\vec{r} - \vec{a}) \sim \left[r^2\vec{p} - 2\vec{r}(\vec{r}\vec{p})\right](\vec{r} - \vec{a})$$

Opening the brackets and using the equality $\vec{a}\vec{p} = 0$ we get:

$$\vec{j}_{\text{outside}}(\vec{r} - \vec{a}) \sim (2\vec{a}\vec{r} - r^2)(\vec{r}\vec{p})$$

which equals zero.

Note no. 2: The skin effect. Finally we need to discuss the applicability limits of the solution obtained above. We have assumed that at each point on the plate, the current density is constant across the entire z-directional intersection of the plate. This requires the skin depth to be much greater than the thickness of the plate. The skin effect emerges due to the fact that for resistive media, the evolution of the magnetic field can be described by a diffusion equation

$$\frac{\partial \vec{B}}{\partial t} = D\frac{\mathrm{d}^2\vec{B}}{\mathrm{d}z^2},$$

where the diffusivity $D = \rho/\mu_0$. This means that for a very fast moving magnet, there is no time for the magnetic field to penetrate into the plate. Since any current is surrounded by a magnetic field according to Ampère's circuital law, if there is no magnetic field, there can be no current, either. Consequently, the current will flow only in a narrow boundary layer of the plate. For a diffusive process, during a characteristic time τ, the penetration depth is estimated as $\sqrt{D\tau}$. With τ estimated as R/v, we get the condition

$$\rho R \gg \mu_0 v \delta^2$$

which is well satisfied for any metallic plates with reasonable orders of magnitude (e.g. $\delta \sim 1\,\text{mm}$, $R \sim 1\,\text{cm}$, $v \sim 1\,\text{m/s}$).

E1 - Magnetic Pendulum – solution

Motivation

In the vibration isolation system of the VIRGO gravitational wave detector (see Fig. 1), repulsive magnets are used to shift the vertical oscillation frequency of the suspension from around $1.5\,\text{Hz}$ to below $0.5\,\text{Hz}$. This improves the sensitivity of the detector to gravitational waves at a frequency of few Hz.

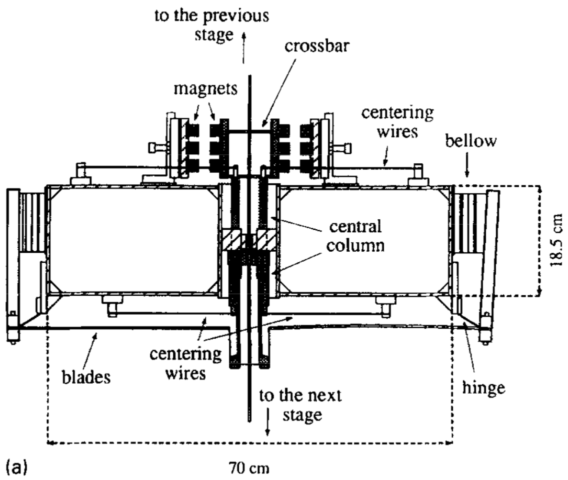


Figure 1: Magnetic anti-spring as part of the mechanical suspension of the VIRGO gravitational wave interferometer. From NIM A 394 (1997) 397-408.

Derivation of modified pendulum frequency (given to students)

The motion of a physical pendulum is constrained to the $y = 0$-plane (one degree of freedom). A magnetic dipole (oriented along the y-axis) is attached to the pendulum such that it is located at the origin when the pendulum is in equilibrium. Two more y-oriented magnetic dipoles are placed on the y-axis at $y = \pm d$. The combined dipole moment of the magnets on the pendulum is j_1, the dipole moments of the external magnets are both j_2.

The magnetic field at position $\vec{r}$ generated by a point dipole $\vec{m}$ at the origin, is:

$$\vec{B}(\vec{m},\vec{r}) = \frac{\mu_0}{4\pi}\cdot\left(\frac{3\vec{r}\cdot(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right) \tag{1}$$

The y-component of the magnetic field generated on the x-axis by a dipole $(0, j_2, 0)$ located at $(0, d, 0)$ is therefore:

$$B_y(d,x) = \frac{\mu_0}{4\pi}\cdot j_2\cdot\frac{2d^2 - x^2}{(x^2+d^2)^{5/2}} \tag{2}$$

The function is symmetric in d, so the two external magnets provide equal contributions on the x-axis. The first terms of a Taylor series around $x = 0$ are

$$B_y(d,x) = \frac{\mu_0}{4\pi}\cdot j_2\cdot\left(\frac{2}{d^3} - \frac{6x^2}{d^5} + \frac{45}{4}\frac{x^4}{d^7} + \mathcal{O}\left(x^6\right)\right) \tag{3}$$

The potential energy of the pendulum as a function of its angle φ is:

$$U(\varphi) = Mgs(1-\cos\varphi) + j_1\cdot 2B_y\left(d, 2\ell\cdot\sin(\varphi/2)\right), \tag{4}$$

where s is the distance from the magnetic pendulum COM to the pivot and ℓ is the distance from the pendulum magnet to the pivot. Neglecting constants and higher order terms:

$$U(\varphi) \approx Mgs\varphi^2/2 + \frac{\mu_0}{4\pi}\cdot j_1\cdot 2j_2\cdot\left(-\frac{6}{d^5}\ell^2\varphi^2\right), \tag{5}$$

$$U(\varphi) \approx \left(Mgs - \frac{6\mu_0}{\pi}\cdot j_1\cdot j_2\cdot\frac{\ell^2}{d^5}\right)\cdot\varphi^2/2, \tag{6}$$

The kinetic energy of the pendulum is $T(\dot\varphi) = I\dot\varphi^2/2$, from $\partial_\varphi U(\varphi) = \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot\varphi}T(\dot\varphi)$ we find the natural frequency

$$\omega^2 = \left(Mgs - \frac{6\mu_0}{\pi}\cdot j_1\cdot j_2\cdot\frac{\ell^2}{d^5}\right)\Big/ I \tag{7}$$

We can write

$$\omega^2 = \omega_0^2 \pm \omega_{mag}^2 \tag{8}$$

with $\omega_0^2 = Mgs/I$ and $\omega_{mag}^2 = \frac{6\mu_0}{I\pi}\cdot j_1\cdot j_2\cdot\frac{\ell^2}{d^5}$, where the plus sign is for attractive external magnet polarity and the minus sign for repulsive external magnet polarity.

Task 1 - Mass of magnets and pendulum body (1.0 pts)

Without scales for weight measurements, there are multiple ways to determine the mass ratio between magnets and pendulum:

A) finding the center of mass of the pendulum with and without magnets (or with magnets in different positions)

B) finding the tilt angle of the pendulum with magnets offset horizontally (very precise by laser angle measurement)

C) measuring the pendulum frequency as a function of vertical magnet position, and using e.g. the position of fastest oscillation

The methods allow very different precision (with B best and C worst typically). Full score is achievable with any method if the numerical result is good.

All pendulums were weighed on an analytical balance. We measured a mass distribution for $M_{\text{pen}} = (44.7\pm1.7)\,\text{g}$ (mean and standard deviation). The mass of the two pendulum magnets agrees well with the specifications $M_{\text{mag}} = (7.68\pm0.01)\,\text{g}$. The nominal mass ratio follows as

$$M/m = 5.82 \pm 0.22. \tag{9}$$

Measurements of twelve pendulum magnets are in agreement with the specified mass of $7.68\,\text{g}$ for one pair. Separately measuring the six pairs reveals a sample standard deviation of only $0.035\,\text{g}$.

Table 1: Physical parameters of pendulum

quantity	value
mass of pendulum body	$M_{\text{pen}} = (44.7 \pm 1.7)$ g
mass of pendulum magnets	$M_{\text{mag}} = (7.68 \pm 0.03)$ g
distance magnets-axis	$\ell = (23.0 \pm 0.1)$ cm
distance COM-axis (no magnets)	$s_0 = (12.0 \pm 0.1)$ cm
distance COM-axis (with magnets)	$s_1 = (13.6 \pm 0.1)$ cm
period without magnets	$T_0 = 2\pi/\omega_0 = (863 \pm 5)$ ms
period with magnets	$T_1 = 2\pi/\omega_1 = (885 \pm 3)$ ms
moment of inertia (with magnets)	$I = (1.38 \pm 0.02)$ g · m^2
dipole moment pendulum magnets	$j_1 = (0.96 \pm 0.01)$ A · m^2
dipole moment external magnet	$j_2 = (2.30 \pm 0.03)$ A · m^2

Since the magnet mass is more consistent between setups than the pendulum mass (and both measurements are fully anticorrelated by the given sum), points are awarded only for the determined magnet mass. The accepted range is chosen large enough to cover the spread in determined magnet mass caused by the variation in total mass relative to the given value of 52.3 g.

A: center of mass method By balancing the pendulum e.g. on a ruler, one can first mark the center of mass without magnets on the pendulum (S), then attach the magnets in a position P as far away from S as possible (e.g. bottom center), and measure the new center of mass (S'). Balance of torques around S' gives

$$M \cdot \overline{SS'} = m \cdot \overline{S'P} \tag{10}$$

With measured values of $\overline{SS'} = (1.6 \pm 0.1)$ cm and $\overline{S'P} = (9.4 \pm 0.1)$ cm one finds:

$$M/m = 5.9 \pm 0.4. \tag{11}$$

The separate masses can be expressed by the sum and ratio:

$$m = \frac{M+m}{1+M/m} = (7.6 \pm 0.5)\,\text{g} \tag{12}$$

$$M = (M+m) \cdot \frac{M/m}{1+M/m} = (44.7 \pm 0.6)\,\text{g}. \tag{13}$$

B: tilt angle method (lighter prototype pendulum) With the magnets placed at a position (x_m, y_m) relative to the pivot (projected onto the pendulum), the pendulum comes to rest at a tilt angle α relative to vertical. This angle can be measured precisely by observing the deflection δy of the laser spot at a screen distance d:

$$\tan \alpha = \delta y/d \tag{14}$$

N.B.: there is no factor two, since the rotation axis of the mirror is parallel to the laser beam.

Calling the angle between vertical and the line through pivot and magnet equilibrium position β, balance of torques becomes:

$$M \cdot s_0 \cdot \sin \alpha = m \cdot \sqrt{x_m^2 + y_m^2} \cdot \sin \beta \tag{15}$$

where

$$\tan(\alpha + \beta) = x_m/y_m \tag{16}$$

The COM distance from the pivot is $s_0 = (11.4 \pm 0.1)$ cm. The best magnet position for a single measurement is close to the edge at the widest part of the pendulum. For $x_m = (7.7 \pm 0.1)$ cm, $y_m = (18.0 \pm 0.1)$ cm and $d = (55 \pm 1)$ cm, we obtained $\delta y = (5.9 \pm 0.1)$ cm. From this we calculate $\alpha = 6.12°$, $\beta = 17.0°$, and $M/m = 4.7 \pm 0.4$.

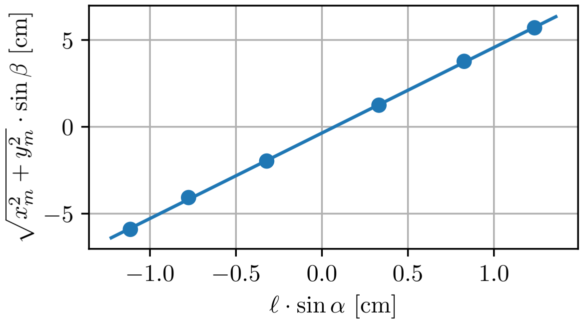


Figure 2: Analysis of tilt angle for several magnet positions.

For higher precision the balance of torques should be graphed for several magnet positions, see Fig. 2. The slope yields $M/m = 4.92 \pm 0.03$ (fit error only). The intercept of (-0.36 ± 0.02) cm can be explained by a failure to notice a 1 mm offset of the COM from the symmetry axis of the pendulum, rotating the coordinate system for β by $1°$.

C: frequency method (lighter prototype pendulum) The pendulum frequency depends on the position of the magnet placed at $(0, y)$ relative to the pivot as follows:

$$\omega^2 = \frac{Ms_0 + my}{I + my^2} \cdot g. \tag{17}$$

This function has a unique maximum ω_{max} at

$$y_{max} = \sqrt{I/M + \left(\frac{Ms_0}{m}\right)^2} - \frac{Ms_0}{m}. \tag{18}$$

Eliminating y_{max} from the expression for ω_{max}, we can solve for M/m:

$$m/M = \frac{4\omega_{max}^2}{\Omega^2} \cdot \left(\frac{\omega_{max}^2}{\omega_0^2} - 1\right) \tag{19}$$

with $\Omega^2 = g/s_0$ and $\omega_1^2 = Ms_0g/I$.

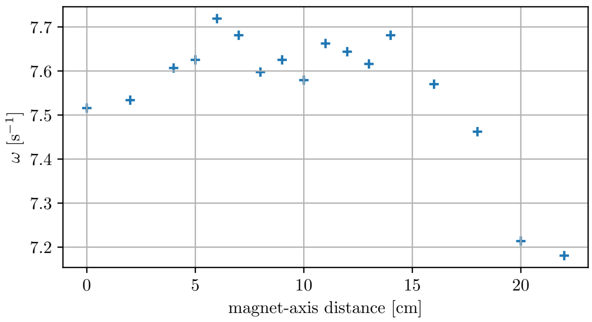


Figure 3: Pendulum frequency as a function of vertical magnet position.

After a scan of different y-values, we find $y_0 = (10 \pm 3)$ cm, $\omega_{max} = (7.65 \pm 0.05)\,\mathrm{s}^{-1}$, $\omega_1 = (7.50 \pm 0.06)\,\mathrm{s}^{-1}$ and $\Omega = (9.28 \pm 0.04)\,\mathrm{s}^{-1}$. We calculate $M/m = 6.7 \ldots 14$.

This method is not precise enough with the relatively high mass ratio of the setup.

variant: frequency with/without magnets There are many ways to extract the mass ratio from pendulum frequencies. A popular method was to compare the frequency with and without magnets, using only the nominal magnet position. The two expressions for the frequencies are

$$I \cdot \omega_0^2 = Mgs_0 \tag{20}$$

and

$$(I + m\ell^2) \cdot \omega_1^2 = (M + m)gs_1 \tag{21}$$

The mass ratio can be expressed as

$$M/m = \frac{\ell^2/g - s_1^2/\omega_1^2}{s_1^2/\omega_1^2 - s_0^2/\omega_0^2} = 6.1 \pm 0.3. \tag{22}$$

E1.1 – Masses		**Points**
A	**Center of mass method**	**0.6**
	Determination of distances $\overline{SS'}$ and $\overline{S'P}$ (0.1 per repetition with different magnet position)	0.4
	expression for mass ratio	0.2
B	**Tilt angle method**	**0.6**
	# of data points. 1:0.1, 2:0.3, ≥ 3: 0.4	0.4
	expression for mass ratio (or equivalent calculations for finding the masses)	0.2
C	**Frequency method**	**0.6**
	measurement of ω_1 and ω_0 (0.1 per repetition with $\geq$ 5 oscillations, or 0.1 per 10 oscillations without repetition)	0.4
	expression for mass ratio	0.2
	Numerical value for M_{mag}	**0.4**
	value inside (7.7 ± 1.0) g	or 0.2
	value inside (7.7 ± 1.5) g	or 0.1
	reasonable estimate of uncertainty	0.2
Total on Masses		**1.0**

Task 2 - Magnetic dipole moments (4.0 pts)

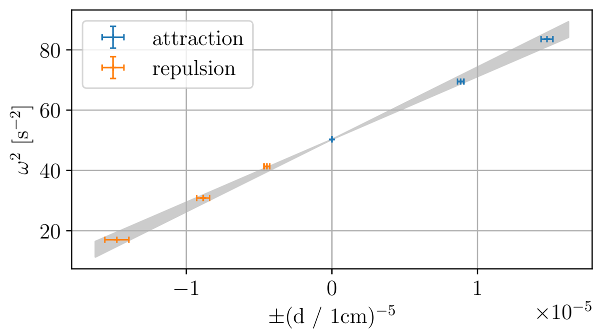


Figure 4: Determination of magnetic frequency shift by external dipole magnets.

a) Data collection The accessible frequency range is maximized by using both possible external dipole orientations. The closest possible distance in attractive orientation is given by stability of the magnets in the rail against sliding towards the pendulum, and is around $d = 7$ cm. In the repulsive mode it is given by the full compensation point near $d = 8.4$ cm, distances down to around 9 cm are practical.

For closer magnet spacings in repulsive mode, the pendulum moves in a double-well potential with an ill-defined behavior around zero amplitude. Eqn. **??** suggests a negative ω^2 and is not valid for oscillations around the off-set minimum. These data points are therefore not useful for this problem. This is specified in the hint that the magnets must be collinear when the pendulum is in equilibrium.

Since one needs to plot ω^2 vs d^{-5}, small d need to be sampled much more finely to spread the data points in d^{-5}. Ideally, one can pre-calculate an equal spacing, e.g. $(0^{-0.2}, 0.25^{-0.2}, 0.5^{-0.2}, 0.75^{-0.2}, 1^{-0.2}) \cdot d_0 =$

$(\infty, 1.32, 1.15, 1.06, 1) \cdot d_0$. The point at infinity is recorded by removing the magnets.

b) Specific magnetization Both attractive and repulsive datasets can be combined in one plot by counting repulsive magnet spacings as negative, absorbing the sign at $\pm\omega^2_{mag}$:

$$\omega^2 = \omega_1^2 + \frac{6\mu_0}{I\pi} \cdot j_1 \cdot j_2 \cdot \frac{\ell^2}{d^5} \tag{23}$$

Plotting ω^2 vs d^{-5}, one can fit a linear relation with slope $k = \frac{6\mu_0 \ell^2}{I\pi} \cdot j_1 \cdot j_2$ and intercept ω_1^2.

The example data in Fig. 4 yield a slope of $k = (2.14 \pm 0.05)\, 10^6 \mathrm{cm}^5/\mathrm{s}^2$. To extract the dipole moment product one must measure the magnet-axis distance $\ell = (23.0 \pm 0.1)\,\mathrm{cm}$ and find the moment of inertia I. This can be extracted from the undisturbed frequency $\omega_1^2 = (M_\text{pend} + M_\text{mag}) g s_1 / I$, with the given total mass and gravitational acceleration, and the COM-axis distance $s_1 = (13.6 \pm 0.1)\,\mathrm{cm}$ (by balancing). One finds:

$$I = \frac{(M_{pend} + M_{mag}) g s_1}{\omega_1^2} = (1.38 \pm 0.02)\,\mathrm{g} \cdot \mathrm{m}^2 \tag{24}$$

The dipole moment product follows as

$$j_1 \cdot j_2 = k \cdot \frac{I\pi}{6\mu_0 \ell^2} = (2.33 \pm 0.05)\,(\mathrm{Am}^2)^2. \tag{25}$$

Since the ratio $j_2/j_1 = 2.4$ is given, the dipole moments separately are

$$j_1 = \sqrt{j_1 \cdot j_2 / 2.4} = (0.98 \pm 0.01)\,\mathrm{Am}^2. \tag{26}$$

$$j_2 = \sqrt{j_1 \cdot j_2 \cdot 2.4} = (2.36 \pm 0.03)\,\mathrm{Am}^2. \tag{27}$$

With the true mass of the pendulum magnets $M_{magnets} = 7.68\,\mathrm{g}$ we find a specific magnetization of

$$j_1 / M_{magnets} = (0.128 \pm 0.030)\,\mathrm{Am}^2/\mathrm{g}. \tag{28}$$

The expected value can be calculated from manufacturer values of remanence ($1.29\,\mathrm{T}$–$1.32\,\mathrm{T}$) and density ($7.4\,\mathrm{g/cm}^3$–$7.5\,\mathrm{g/cm}^3$) of NdFeB–N42:

$$\frac{B_r}{\mu_0 \cdot \rho} = 0.137\,\mathrm{Am}^2/\mathrm{g} - 0.142\,\mathrm{Am}^2/\mathrm{g}. \tag{29}$$

The 10% reduction of the observed specific magnetization may be explained by edge effects in the small magnets, where magnetization may not be homogeneous. There is a hint for edge effects also from the magnetic moment ratio between external and pendulum dipole moments, 2.4 measured with a magnetometer, higher than the volume ratio, 2.2 measured without the coatings.

The data collection and linear regression was performed for five different setups, the slope was consistent within $\pm 5\%$ (explained by observed magnetic moment variations) while the offset (squared frequencies) was stable within $\pm 1\%$. The accepted numerical ranges cover this variation.

E1.2 - Magnetic Dipole Moments		**Points**
a)	**Frequency measurements**	**2.0**
	Both attraction and repulsion used	0.3
	Closest d setting $\leq 9\,\mathrm{cm}$ (0.1 if $\leq 10\,\mathrm{cm}$)	0.3
	Denser spacing for small $\lvert d \rvert$	0.3
	0.1 per different d-setting (incl. ∞)	0.6
	0.1 per 10 oscillations per d value	0.5
b)	**Specific magnetization**	**2.0**
	Graph of ω^2 vs d^{-5} or ω vs $d^{-5/2}$ or $\log \lvert \omega^2 - \omega_1^2 \rvert$ vs $\log d$	(1.0)
	correctly enter data points (0.1 each)	0.5
	correct axis labels and ticks	0.1
	data covers $\geq \frac{1}{2}$ page	0.1
	trend line drawn	0.1
	slope read	0.1
	slope error estimated	0.1
	expression for determining I	0.3
	expression for determining $j_1 \cdot j_2$	0.2
	expression for determining j_1	0.2
	Numerical value for j_1/M_{mag} = $0.125\,\mathrm{Am}^2/\mathrm{g}$, or alternatively $j_1 \cdot j_2 = 2.2\,(\mathrm{Am}^2)^2$ (0.3 within $\pm 10\%$, 0.2 within $\pm 20\%$, 0.1 within $\pm 30\%$)	0.3
Total on Magnetic Dipole Moments		**4.0**

If, instead of a graphical analysis, a slope was calculated from two data points, we award 0.2/0.5 points for "entering data", no points for axis, space and trend line (total loss: 0.6 points). For the rest we apply the normal grading scheme.

Task 3 - Unknown external magnets (3.0 pts)

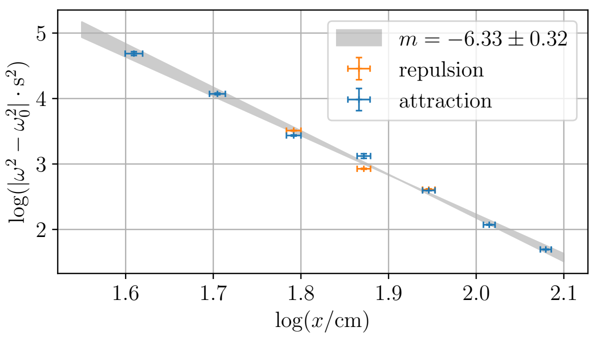


Figure 5: Determination of unknown magnet power law.

a) Data collection By experimentation one can find that the unknown magnets are symmetric under a $180°$ rotation, not anti-symmetric as the dipoles. It is still possible to obtain both attraction and repulsion, either by swapping the unknown magnets (which are given to be oppositely magnetized to each other), or by flipping the pendulum magnets. Flipping the pendulum magnets may require a new measurement of ω_1 if the magnet position is not exactly the same.

As the overall field strength is weaker, distances down to $d = 5\,\text{cm}$ are possible in attractive mode, as well as about $d = 5.5\,\text{cm}$ in repulsive mode. For equal spacing in log-scale one can choose d values increasing by roughly the same factor between following measurements.

Attractive and repulsive mode can not be combined to increase precision, but can serve as independent measurements of the absolute value of magnetic frequency shift at a given separation. Only the attractive mode can be selected for its higher accessible d-range and overall faster frequencies (higher relative precision).

b) Power law Plotting the absolute value of the squared magnetic frequency shift $|\omega^2 - \omega_1^2|$ versus the magnet distance on double-logarithmic axes (Fig. 5), one can fit a linear relation with slope $a = -6.33 \pm 0.32$, indicating a power law $\omega_{mag}^2 \propto d^{-6}$.

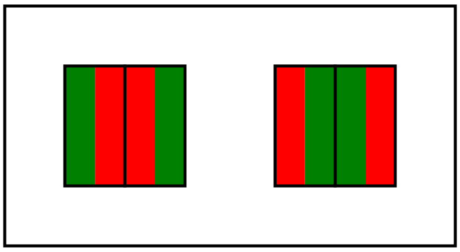


Figure 6: Dipole configuration of unknown external (quadrupole) magnets.

c) Dipole configuration Since the exponent of the magnetic frequency shift is one higher than for the dipole magnets, we can conclude that also the exponent of the magnetic field as a function of distance is one higher than for the dipole law, $B \propto r^{-4}$. (In fact the squared magnetic frequency shift follows the curvature of the magnetic field, the derivatives add two to the power law exponent.)

We are looking for a magnetic quadrupole configuration. This can be created by an arrangement of equal and opposite dipole magnets (canceling the total dipole moment of the arrangement). The simplest solution consistent with the package (see Fig. 6) consists of two coaxial and opposite cylindrical dipole magnets in each unknown external magnet, one with north poles facing out, one with south poles facing out.

Another valid solution is made of cylinders with radial magnetization (i.e. south poles towards the symmetry axis, north poles outwards, and vice versa). This would be harder to build but yields magnets indistinguishable from the actual ones to leading order.

Other possibilities of creating a quadrupole out of opposite dipoles (e.g. with dipole offset not parallel to the magnetic moments) are excluded because of the cylindrical symmetry of the unknown external magnets (this can be tested e.g. by rotation of the unknown magnet in a dipole field).

E1.3 - Unknown external magnets		**Points**
a)	**Data collection**	**1.0**
	Closest d setting $\leq 6\,\text{cm}$	0.2
	Denser spacing for small $\lvert d \rvert$	0.2
	0.1 per 2 different d-settings	0.3
	0.1 per 10 oscillations per d value	0.3
b)	**Power Law**	**1.5**
	Graph of ln $\left(\lvert\omega^2 - \omega_1^2\rvert\right)$ vs $\ln(d)$	(1.0)
	correctly enter data points (0.1 each)	0.5
	correct axis labels and ticks	0.1
	data covers $\geq \frac{1}{2}$ page	0.1
	trend line drawn	0.1
	slope read	0.1
	slope error estimated	0.1
	Exponent value (-6 ± 0.5: 0.5, -6 ± 1: 0.2)	0.5
c)	**Dipole configuration**	**0.5**
	Sketch of possible dipole configuration	0.3
	justification (0.1 each for magnetic symmetry under reversal, locating the opposite pole in the center, magnetic cylindrical symmetry, or cancellation of dipole moments)	0.2
Total on Unknown external magnets		**3.0**

If, instead of a graphical analysis, a slope was calculated from two data points, we apply the grading scheme of 1.2b).

In 1.3c), if no points were awarded for justification, also no points are given for the correct configuration.

Task 4 - Nonlinear pendulum (2.0 pts)

a) Compensation Distance A precise way of finding the compensation distance is from the analysis of E1.2. We are interested in the x-intercept of the regression line, with slope $k = (2.14 \pm 0.05)\,10^6\text{cm}^5/\text{s}^2$ and y-intercept $\omega_1^2 = (50.3 \pm 0.3)\,\text{s}^{-2}$. We find $d_{comp}^5 = k/\omega_1^2$, or $d_{comp} = (8.43 \pm 0.05)\,\text{cm}$.

Another possibility is to manually approach the compensation point and carefully check for the absence of two separate minima.

A less useful method is the recording of the angular separation $\Delta\varphi$ of the split minima for $d < d_{comp}$ as a function of d. This relation is strongly curved and therefore not suited for a precise extrapolation to $\Delta\varphi = 0$.

b) Period Power Law Near the point of full compensation of the quadratic terms of gravitational and magnetic potential, an interesting dependence of pendulum frequency on amplitude can be observed.

Fig. 7 shows numerical calculations of the period for different magnet spacings close to full compensation (black) and without external magnets (red).

For experimentally achievable configurations, three ranges can be distinguished:

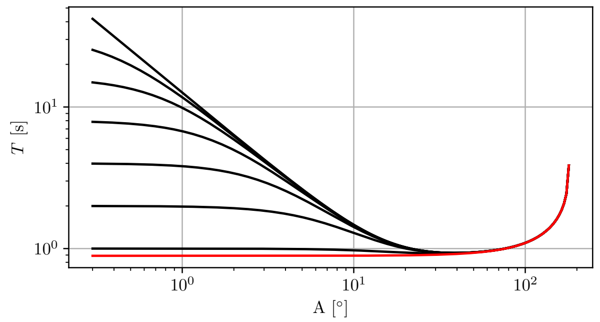


Figure 7: Numerical calculation of pendulum period versus amplitude, for different degrees of compensation.

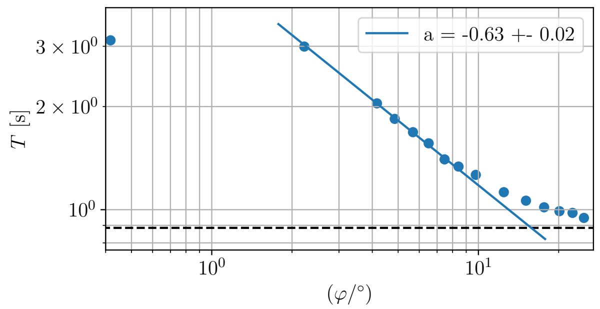


Figure 8: Measured pendulum period versus amplitude.

- for small amplitudes the period is constant, given by the small remaining quadratic potential term from imperfect compensation
- in an intermediate amplitude range, the magnetic quartic potential dominates, leading to a theoretical dependence $T(\varphi) \propto \varphi^{-1}$ with amplitude φ.
- for large amplitudes, the influence of the localized magnetic potential is small. The period approaches that of the undisturbed pendulum.

The extent of the relevant intermediate amplitude range depends on the quality of compensation, for example quantified by the ratio $T_0/T(\varphi = 0)$ of the compensated small-amplitude period to the small-amplitude period without magnets.

Typically the theoretical exponent of -1 is not reached in experiment, because the quartic potential does not fully dominate for any amplitude region. The expected maximum slope of the power law can be calculated numerically as a function of $T_0/T(\varphi = 0)$, see Fig. 9. (The exponent averaged over a factor of 2 in amplitude around the maximum is very close to this maximum value.)

The data shown in Fig. 8 was acquired with a compensation quality corresponding to $T_0/T(\varphi = 0) = 4.08$. This results in an expected maximum exponent of -0.645 (-0.63 when averaged over a factor 2 in amplitude). This agrees well with a linear regression around the steepest part of the amplitude-period relation.

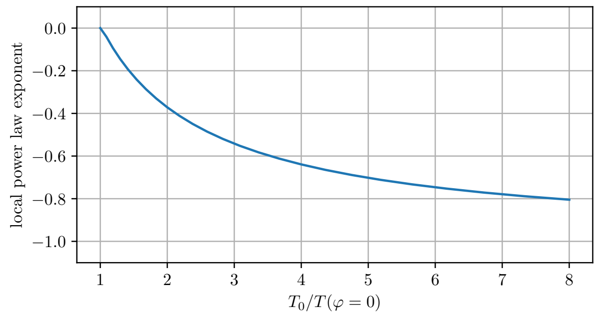


Figure 9: Maximum power law slope as a function of "degree of compensation".

E1.4 – Nonlinear Pendulum		**Points**
a)	**Cancellation point**	**0.5**
	Numerical value for $d_{comp} = (83.3 \pm 2.0)$ mm (0.2 within 4 mm)	0.5
b)	**Amplitude dependence**	**1.5**
	0.1 per 2 amplitude values	0.4
	0.1 per oscillation at each amplitude value	0.3
	double-logarithmic graph	0.5
	discussion (0.1 each for plateau at small amplitudes, power law at intermediate amplitudes, or plateau at large amplitudes)	0.3
Total on Nonlinear Pendulum		**2.0**

E2: Optical Black Box - Solution

Task E2.1 - Central element (∼0.3 pts)

To find out what element is located at the center of the box, we systematically beam into the box through all 4 ports (labeled A, B, C, D). We take note of where light exits. We deduce for the four options given:

- **no element**: Can be excluded because in that case we would only see light exiting from opposite ports but we can clearly see a signal around corners, e.g. exiting port B when beaming in through port A.
- **fully reflective mirror (both sides)**: This option we can also exclude since it would not allow for direct transmission along at least one of the optical axes. However, we can clearly see light passing from A to C and B to D (for some input polarizations).
- **regular-triangle-shaped prism:** A regular-triangle-shaped prism has a $60°$ angle between all surfaces. This configuration will always deflect a beam out of the optical axis, but as we clearly see that a central beam remains straight, the prism can be excluded.
- **semi-transparent mirror**: For every input beam, the box produces two significant output beams. This behaviour is well explained by a semi-transparent mirror. In fact we used a $2\,\mathrm{mm}$ thin acrylic glass plate with a semi-transparent window foil on one side.

To find out the orientation of the beam splitter (i.e. the orientation of the partially reflective surface), notice:

- To connect the two (perpendicular) optical axes of the black box, it needs to sit under a $45°$ angle with respect to both.
- Since ports A and B as well as C and D are connected, the partially reflective surface runs diagonally from the corner between D and A to the corner of B and C. (see Fig. 10)

E2.1 Central Element		Points
	Systematic observation of light splitting (automatically given if correct identification)	0.1
	Correct identification of semi-transparent mirror	0.1
	Correct deduction: Orientation of the mirror	0.1
Total on E2.1		**0.3**

Task E2.2 - Port elements (∼2.2 pts)

We systematically beam in through all four ports and write down the observed output for the remaining three ports. Also, we pay attention to effects resulting from a varying input polarization (by rotating the laser diode) and divergence and convergence of the beam. This way, we obtain the observation matrix Table 2.

From this, we conclude:

- **Polarizer at port D:** Whenever port D is involved, there is a strong polarization-dependent behavior and there is no polarization effect in all combinations not involving D. Therefore, D is a polarizer.
- **Diffraction Grating at port C:** There are always three beams coming out of port C. Therefore, there must be a diffraction grating at C. *Note: When beaming in through port C, the higher diffraction orders are clipped at the other ports such that there is only one beam exiting.*
- **Convex lens at port A:** Beaming along the axis connecting ports C-A, we can clearly see a focus outside the box - which can only come from the element at port A – and therefore, there must be a convex lens.
- **Concave lens at port B:** Along the axis B-D, we obtain a diverging beam. This could originate from a concave lens, or from a convex lens with a focus inside the box at B. The beam divergence is appar-

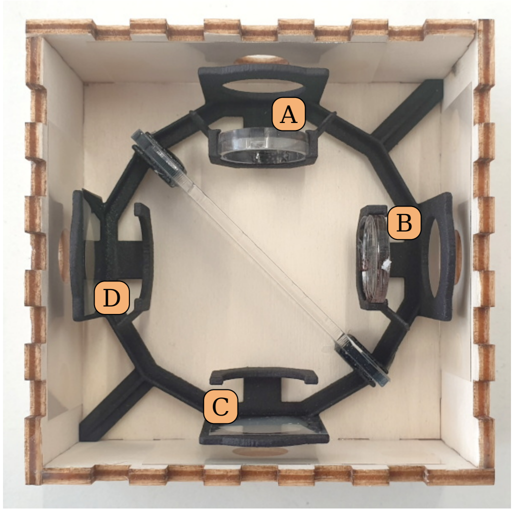


Figure 10: View inside the box

Table 2: Observed Light Properties when beaming in through one port (rows) and exiting through another (columns)

In ↓	Exit A	Exit B	Exit C	Exit D
A	-	focused beam, focus close to box	three focused beams, focus close to box	weak reflection, depends on input polarization
B	bright, focused beam, focus far away	-	three beams, very weak	diverging beam, depends on input polarization, intensity can be reduced to zero
C	focused beam, focus roughly 5cm away from box	very weak, diverging beam	-	collimated beam, depends on input polarization but intensity can not be reduced to zero
D	very weak, focused beam	diverging beam, depends on input polarization, intensity can almost be reduced to zero	three, collimated beams, depends on input polarization, intensity can almost be reduced to zero	-

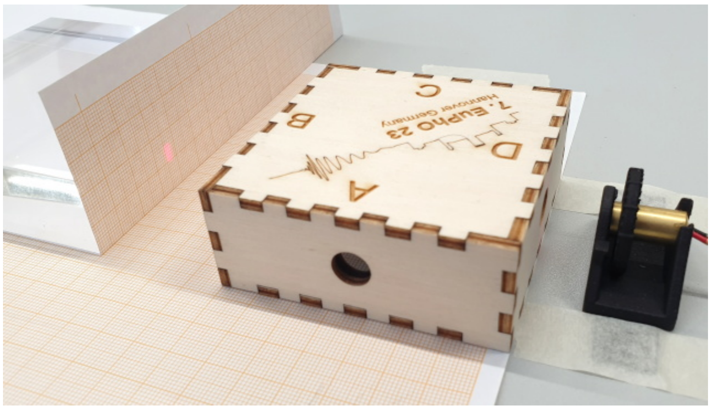


Figure 11: Setup to measure the diverging beam after the concave lens.

ently so flat that it does not converge to a focal spot within the area of B. Thus we conclude that at B, there is a concave lens.

E2.2 Elements		**Points**
a)	Recognizing or using the laser diode as a linearly polarized source of light	0.2
b)	**Observations and solution**	**2.0**
	A: observation / reasoning (beam focus)	0.2
	A: result focussing lens	0.3
	B: observation / reasoning (diverging beam)	0.2
	B: result concave lens	0.3
	C: observation / reasoning (separate spots)	0.2
	C: result grating	0.3
	D: observation / reasoning (rotation dependence)	0.2
	D: result polarizer	0.3
Total on E2.2		**2.2**

Task E2.3 - Properties (∼7.5 pts)

Now, we systematically conduct measurements to obtain the desired values of the four elements. Note that each window contains a protective glass plate of thickness 0.13 mm, so the elements appear 0.04 mm closer to the box edge than they actually are. For true positions we use corrected values.

a) Convex lens behind port A (position and focal length) We start by verifying the beam exiting the laser being collimated with a constant spot size $w_0 \approx (3.8 \pm 0.2)$ mm (may differ for each laser between 3 mm and 6 mm). The possibility of a collimated beam is apparent from the Rayleigh range z_R (not required from students):

$$z_R = \frac{\pi w_0^2}{4\lambda} \approx 16\,\text{m} \tag{30}$$

Realistically, the optical paths used in this setup will be below 50 cm, therefore, the widening of the Gaussian beam can be neglected in comparison to our measurement precision of the beam diameter.

Beaming in through port C (where we located the diffraction grating, whose zeroth order has the same beam profile as the incoming beam), we now measure the spot size as a function of distance from port A. We assign negative values to the spot diameters measured *after* the focus which we can roughly locate by eye to be at around 5 cm away from the box, to use a linear fit function for the beam envelope. The measured values are:

x [cm]	w [mm]
1	2.6
2	1.9
3	1.2
4	0.7
6	-1.0
7	-1.7
8	-2.1
9	-2.8
10	-3.2
11	-3.7
12	-4.3
13	-4.9
14	-5.3
15	-5.8

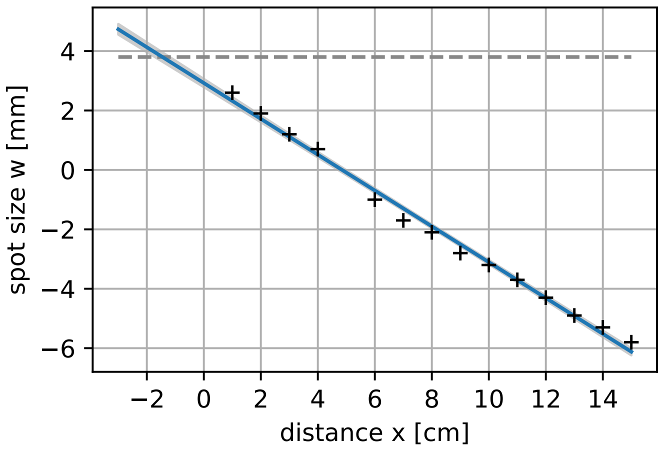


Figure 12: Linear fit of the spot size after the convex lens vs. distance from box edge

The spot size can therefore be described by a linear equation:

$$w(x) = w_0 - \frac{w_0}{f}(x - x_0) = -\frac{w_0}{f}x + w_0\left(\frac{x_0}{f} + 1\right). \tag{31}$$

Using the data plotted in Fig. 12, we obtain a focal length of

$$f = -\frac{w_0}{w_0/f} = \frac{3.8\,\text{mm}}{0.0603} = (6.3 \pm 0.2)\,\text{cm} \tag{32}$$

The true focal length is

$$f_{+,\text{true}} = +6.5\,\text{cm} \tag{33}$$

The position of the lens is found at the spot, where the converging beam diameter is equal to that of the

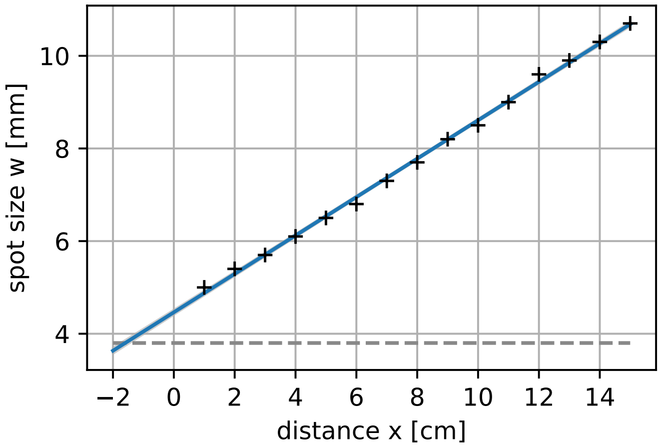


Figure 13: Linear fit of the spot size after the concave lens vs. distance from box edge

original collimated beam (cutting the dashed line in Fig. 12), which is $-1.7\,\mathrm{cm}$ from the edge at the box (while the edge is $3.8\,\mathrm{cm}$ from the center), thus

$$x_+ = 3.8\,\mathrm{cm} + x_0 = (2.1 \pm 0.3)\,\mathrm{cm} \tag{34}$$

The true position is

$$x_{+,\mathrm{true}} = 2.2\,\mathrm{cm} \tag{35}$$

from the center of the box.

b) Concave lens behind port B (position and focal length) To determine the position and focal length of the concave lens, we measure the size of the diverging beam at different positions. The position of the lens is located where the beam diameter would coincide with the original collimated beam.

We use the concave lens port B as an output, and insert the beam at port D, where the polarizer will not disturb the beam divergence. To measure the beam size at different positions, we tape a piece of paper with millimeter-scale onto the glass block, and mark the beam edges with a pencil. This allows us to measure the width with sub-mm precision (around $0.2\,\mathrm{mm}$ root-mean-squared).

The following recorded beam widths are shown in Fig. 13:

x [cm]	w [mm]
1	5.0
2	5.4
3	5.7
4	6.1
5	6.5
6	6.8
7	7.3
8	7.7
9	8.2
10	8.5
11	9.0
12	9.6
13	9.9
14	10.3
15	10.7

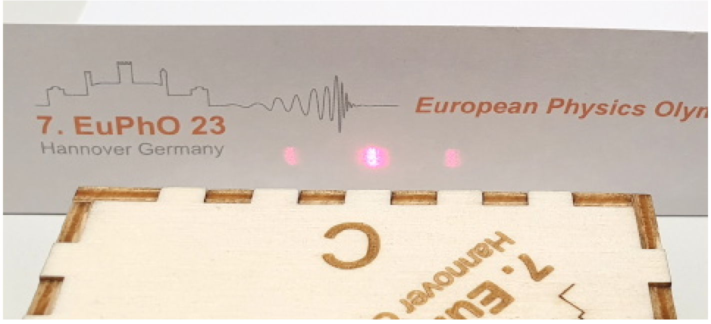


Figure 14: Setup for determining the grating distance and position.

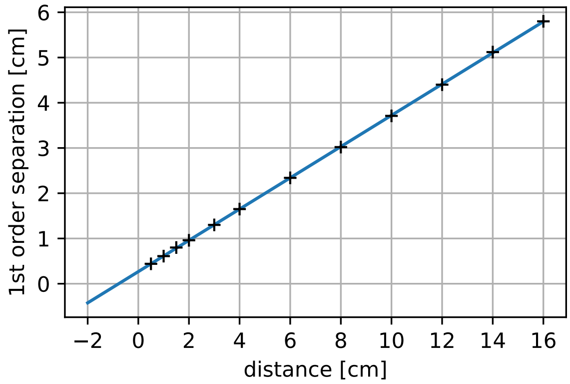


Figure 15: Linear fit of the first order separation over distance

From the fit, we determine the slope

$$w' = 0.0415 \pm 0.0005 \tag{36}$$

and the point $x_0 = (-1.6 \pm 0.1)\,\mathrm{cm}$, where the envelope cuts the original beam width $w_0 = 3.8\,\mathrm{mm}$

$$x_- = 3.8\,\mathrm{cm} + x_0 = (2.2 \pm 0.1)\,\mathrm{cm} \tag{37}$$

which is our estimate for the position of the concave lens. The true position is actually

$$x_{-,\mathrm{true}} = 2.1\,\mathrm{cm} \tag{38}$$

from the center of the box.

The focal length is determined by

$$f_- = -\frac{w_0}{w'} = (-9.2 \pm 0.3)\,\mathrm{cm} \tag{39}$$

The true focal length of the concave lens has been measured to be:

$$f_{-,\mathrm{true}} = (-9.2 \pm 0.15), \mathrm{cm} \tag{40}$$

Alternative method: For the measurement of the concave lens, it is also possible to move the box strictly sideways (e.g. along a fixed ruler) and measure the displacement of the beam at large distances. The possible displacement is around $7\,\mathrm{mm}$, and precisions around $5\,\%$ can be expected.

c) Diffraction grating behind port C (position, rotation and pitch) The grating produces two sharp side-beams of 1st and −1st order in horizontal direction. Therefore it is a linear grating with vertical lines (orientation). The line separation (pitch) can be determined by measuring the angle of the created beams relative to the 0th order transmission.

To avoid distraction from the lenses, we choose to enter through the polarizer (D) and exit through the grating at port C, and turn the laser to maximum transmission through the polarizer. By measuring the transversal beam separation at several distances from the box, we can fit the progression linearly, and obtain both the diverging angle and the offset position of the grating.

The diffraction angles α for each order n fulfill the relation

$$d \cdot \sin\alpha = n \cdot \lambda \tag{41}$$

where λ is the laser wavelength of $650\,\text{nm}$.

We measured the following position values, where we have a higher point density near the box for a precise determination of the position, and a wide range of distances for a precise determination of the beam angle

x [cm]	y [cm]
0.5	0.44
1	0.61
1.5	0.80
2	0.96
3	1.3
4	1.65
6	2.34
8	3.02
10	3.71
12	4.4
14	5.12
16	5.8

which are plotted in Figure 15.

A linear fit yields the slope $y' = 0.3454 \pm 0.0005$ and a value of $y = 0$ at $x_0 = (-0.77 \pm 0.01)\,\text{cm}$ and $\alpha = \arctan y' = 19.1°$. Therefore we determine the pitch as

$$d = \lambda / \sin\alpha = (1.99 \pm 0.02)\,\mu\text{m} \tag{42}$$

The error of this is dominated by the uncertainty of the laser wavelength ($5\,\text{nm}/650\,\text{nm} = 0.8\,\%$), while the measured slope only has an uncertainty of $0.0005/0.3454 = 0.14\,\%$.

$$(\text{true value } 2\,\mu\text{m}) \tag{43}$$

The position of the grating inside the box is where the fitted line cuts the x-axis. The main uncertainty for this stems from the placement accuracy of box and screen with roughly $1\,\text{mm}$,

$$x_g = 3.8\,\text{cm} + x_0 = (3.0 \pm 0.1)\,\text{cm} \tag{44}$$

from the outer edge of the box. The true value is $x_{g,\text{true}} = 3.12\,\text{cm}$.

Figure 16: Determination of the laser polarization using Brewster reflection. The reflected beam will vanish at horizontal polarization.

d) Polarizer behind port D (rotation angle) To characterize the polarizer, we need to know the precise polarization of the laser beam. The laser can be characterized with the available acrylic glass block, using reflection from the surface under an angle. In particular, there is the Brewster angle, at which the incident light will be fully separated into two orthogonally polarized components. This Brewster angle can be computed from the given optical density, but more easily it can be probed by minimizing the intensity of a reflected beam.

The students may set the incident angle on a vertical glass surface to the Brewster angle, and simultaneously turn the laser around its optical axis to yield zero reflection (Fig. 16). This is the point where the laser is purely horizontally polarized, and may be marked on the turning wheel.

Now, to probe the polarizer, the laser is sent into the polarizer port (D) as an input. It should not be used as an output, because the central beam splitter might be partially polarizing and thereby disturb the measurement.

Subsequently, the angle of the polarizer is found by turning the laser around its optical axis until the transmission is minimized near zero (The minimum can be found more precisely than the maximum, because the relative intensity change remains large). This angle is noted relative to the angle of vertical laser polarization, and marks the direction of maximum suppression. Thus, the transmitting direction of the polarizer is $90°$ from the measured angle.

The polarization angle is found to be $65°$ from the vertical axis. we do not consider the orientation with respect to mirroring, only the angle from the horizontal axis.

As the measurement of the point of minimal transmission is quick, but not very precise, a good strategy is to take two measurements $180°$ apart and average their results.

E2.3 Properties		**Points**
a)	**Convex lens at A**	**1.9**
	Measurement idea and linearization: beam diameter as function of distance	0.2
	Data collection, at least 10 data points over 15 cm, in case of less: min(0.05 pts × number of measurements, s [cm]/30 pts for data range s, 0.5 pts total). (Alternatively located the focal spot quantitatively: 0.2 pts)	0.5
	Diagram and fit, alternatively analytic	0.6
	Result for $f = 6.5$ cm (± 0.5 cm, half for ± 2 cm)	0.2
	Result for $x_+ = 2.2$ cm (± 0.3 cm, half for ± 0.6 cm)	0.2
	Error propagation and estimation	0.2
b)	**Concave lens at B**	**1.9**
	Measurement idea and linearization: beam diameter as function of distance	0.2
	Data collection, at least 10 data points over 15 cm, in case of less: min(0.05 pts × number of measurements, s [cm]/30 pts for data range s, 0.5 pts total).	0.5
	Diagram and fit, alternatively analytic	0.6
	Result for $f = -9.2$ cm (± 1 cm, half for ± 2 cm)	0.2
	Result for $x_- = 2.2$ cm (± 0.3 cm, half for ± 0.6 cm)	0.2
	Error propagation and estimation	0.2
c)	**Diffraction grating at C**	**1.9**
	Correct pattern orientation (vertical lines)	0.2
	Measurement idea and linearization: diffraction order separation as function of distance	0.2
	Data collection, at least 3 data points over 15 cm. In case of less: min(0.2 pts × number of measurements, s [cm]/30 pts for data range s, 0.5 pts total).	0.5
	Diagram and fit, alternatively analytic	0.4
	Result for $d = 2\,\mu$m ($\pm 0.05\,\mu$m, half for $\pm 0.1\,\mu$m)	0.2
	Result for $x_g = 3.12$ cm (± 0.2 cm, half for ± 0.4 cm)	0.2
	Error propagation and estimation	0.2
d)	**Polarizer at D**	**1.8**
	Use Brewster angle configuration to determine laser polarization (0.1 for idea to use reflections from a dielectric to investigate polarization)	0.4
	When reflected beam intensity is (close to) zero, then horizontally polarized	0.2
	Noticing that the behavior is different when D is used as an out- vs. input.	0.2
	Use port D as input, not as output (and mention that)	0.2
	Use min, not max transmission	0.2
	Result $\lvert\alpha\rvert = 65^\circ$ 0.3pts for $\pm 5^\circ$ (0.2pts for $\pm 9^\circ$) and 0.1pts for orientation closer to horizontal axis. (0pts if there was no plausible way to calibrate the polarization)	0.4
	Error estimation	0.2
Total on E2.2		**7.5**