OBOF

Brazilian Online Physics Olympiad

2nd Phase - October 26th and 27th, 2025

Name: ____________________________

Grade: ____________________________

César Lattes
Open Level
English

Exam Instructions

I. This exam consists of **3** questions.

II. The maximum duration of this exam is **four hours**. In addition to the exam time, **5 minutes** will be given for filling out the online answer sheet.

III. The exam must be taken individually and discussing the solutions to the questions is not allowed during the exam period **October 26th and 27th, 2025**.

IV. If necessary, and unless indicated otherwise, use: gravitational acceleration on Earth's surface $g = 10\,\mathrm{m/s^2}$; specific heat of liquid water $c_a = 1\,\mathrm{cal/(g\,{}^\circ C)}$; latent heat of fusion of ice $L = 80\,\mathrm{cal/g}$; $1\,\mathrm{cal} = 4.2\,\mathrm{J}$; density of liquid water $\rho = 1.0\,\mathrm{g/cm^3}$; Wien's constant $b = 2.898 \times 10^{-3}\ \mathrm{m \cdot K}$; Planck's constant $h = 6.63 \times 10^{-34}\ \mathrm{J \cdot s}$; speed of light $c = 3 \times 10^8\ \mathrm{m/s}$; energy of hydrogen in the ground state $E_n = -\dfrac{13.6}{n^2}\ \mathrm{eV}$; $1\ \mathrm{eV} = 1.6 \times 10^{-19}\ \mathrm{J}$.

Support:

Trivia:

Cesare Mansueto Giulio Lattes, better known as César Lattes (Curitiba, July 11, 1924 — Campinas, March 8, 2005), was a Brazilian physicist, co-discoverer of the pion (π-meson), a discovery that led to the 1950 Nobel Prize in Physics being awarded to Cecil Frank Powell, the research leader. Lattes is one of Brazil's most distinguished physicists, and his work was fundamental to the development of atomic physics in the country. He was also a great leader in the Brazilian scientific community and one of the main figures responsible for the creation of the National Council for Scientific and Technological Development (CNPq).

Question 1: Thermal Shield - 10 points[1]

A future mission to Mars plans to use a new engine that requires its fuel to be kept at an extremely low temperature, below its critical boiling point, T_c. To prevent the fuel from evaporating, the spherical tank storing it is surrounded by a thermal insulation system composed of N thin, concentric spherical shields, separated by vacuum. To simplify the analysis, consider that the areas of all shields and the tank are approximately equal to A. The outermost shield is exposed to a constant solar energy flux F_0.

The tank itself generates a small amount of heat, P_{int}, due to monitoring equipment. The total thermal load (heat from outside plus internally generated) is removed by an active cryocooler, which has a maximum refrigeration capacity of P_{max}. The insulation design must be efficient enough so that the total thermal load does not exceed this capacity. For the entire analysis, consider the system in thermal equilibrium, the temperature of deep space as zero, and the Stefan-Boltzmann constant as σ.

Part A: The Ideal Model (Black Bodies) - 4 points

In this part, as a first approximation, we will model the system as if all surfaces (the tank and the N shields) were **perfect black bodies**, with emissivity and absorptivity equal to 1.

- **(a)** **0.6 points** In equilibrium, a constant net heat flux, Q_{net}, passes through each vacuum layer. Write an expression for this flux between shield i (at temperature T_i) and the adjacent shield $i+1$ (at temperature T_{i+1}).
- **(b)** **1.4 points** Show that the relationship between the temperature of the first shield (T_1) and the tank (T_{tanque}) can be written as a function of N and Q_{net}.
- **(c)** **2 points** Use the energy balance on the first shield (the outer boundary condition) and the thermal load constraint on the tank (the inner boundary condition) to find an expression for the minimum number of shields, N_{ideal}, required to maintain the tank at $T_{\text{tanque}} \leq T_c$.

Part B: The Realistic Model (Gray Surfaces) - 6 points

In reality, the materials used for insulation are highly reflective. Now consider that all surfaces have a constant emissivity ϵ much less than 1.

- **(d)** **3 points** First, derive an expression for the net energy flux per unit time between two large parallel surfaces with the same area A and emissivity ϵ, maintained at temperatures T_1 and T_2.
- **(e)** **2 points** Using the result from the previous item, find the new expression for the minimum number of shields, N_{real}, as a function of ϵ and the other variables of the problem.

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[1] By Lucas Praça

(f) **1 point** Compare your expressions for N_{real} and N_{ideal}. For a value of $\epsilon \ll 1$, which of the two numbers of shields is larger? Physically justify why the use of low-emissivity materials is such a crucial factor for thermal insulation design in space missions.

Part A: The Ideal Model (Black Bodies)

(a) For black bodies ($\epsilon = 1$), the power radiated by a surface of area A and temperature T is given by the Stefan-Boltzmann Law, $P = \sigma A T^4$. The net heat flux, Q_{net}, between shield i (temperature T_i) and the adjacent shield $i+1$ (temperature T_{i+1}) is the difference between the power each radiates toward the other.

$$Q_{\text{net}} = P_i - P_{i+1}$$
$$\boldsymbol{Q_{\text{net}} = \sigma A(T_i^4 - T_{i+1}^4)}$$

(b) From the result of item (a), we can isolate the difference of the temperature terms: $T_i^4 - T_{i+1}^4 = \frac{Q_{\text{net}}}{\sigma A}$. We write this relationship for each of the N insulation layers, from the first shield to the tank:

$$T_1^4 - T_2^4 = \frac{Q_{\text{net}}}{\sigma A}$$
$$T_2^4 - T_3^4 = \frac{Q_{\text{net}}}{\sigma A}$$
$$\vdots$$
$$T_N^4 - T_{\text{tanque}}^4 = \frac{Q_{\text{net}}}{\sigma A}$$

Summing these N equations, the intermediate terms $(T_2^4, T_3^4, \dots, T_N^4)$ cancel out in a telescopic sum:

$$(T_1^4 - T_2^4) + (T_2^4 - T_3^4) + \cdots + (T_N^4 - T_{\text{tanque}}^4) = \sum_{k=1}^{N} \frac{Q_{\text{net}}}{\sigma A}$$
$$\boldsymbol{T_1^4 - T_{\text{tanque}}^4 = \frac{N \cdot Q_{\text{net}}}{\sigma A}}$$

(c) We apply the outer and inner boundary conditions.

Outer Boundary (Shield 1): The energy balance on the first shield equates the power absorbed from the Sun to the power emitted to space (out) and to shield 2 (in).

$$P_{\text{abs}} = P_{\text{emit}}$$
$$AF_0 = \sigma A T_1^4 + Q_{\text{net}}$$
$$\implies T_1^4 = \frac{F_0}{\sigma} - \frac{Q_{\text{net}}}{\sigma A} \quad (1)$$

Inner Boundary (Tank): The total thermal load on the tank ($Q_{\text{net}} + P_{\text{int}}$) must be equal to the maximum refrigeration capacity (P_{max}) for the limit case.

$$Q_{\text{net}} + P_{\text{int}} = P_{\text{max}} \implies Q_{\text{net}} = P_{\text{max}} - P_{\text{int}} \quad (2)$$

Substituting (1) into the equation from item (b), we have:

$$\left(\frac{F_0}{\sigma} - \frac{Q_{\text{net}}}{\sigma A}\right) - T_{\text{tanque}}^4 = \frac{N \cdot Q_{\text{net}}}{\sigma A}$$
$$AF_0 - Q_{\text{net}} - \sigma A T_{\text{tanque}}^4 = N \cdot Q_{\text{net}}$$
$$AF_0 - \sigma A T_{\text{tanque}}^4 = (N+1) Q_{\text{net}}$$

Imposing the conditions $T_{\text{tanque}} = T_c$ and the expression for Q_{net} from (2):

$$A(F_0 - \sigma T_c^4) = (N_{\text{ideal}} + 1)(P_{\max} - P_{\text{int}})$$
$$\boldsymbol{N_{\text{ideal}} = \frac{A(F_0 - \sigma T_c^4)}{P_{\max} - P_{\text{int}}} - 1}$$

Part B: The Realistic Model (Gray Surfaces)

(d) The power emitted by surface 1 is $P_1 = \epsilon \sigma A T_1^4$. Surface 2 absorbs a fraction ϵ and reflects $(1-\epsilon)$. The reflected energy returns, is reflected again, and so on. The total power from 1 absorbed by 2 is the sum of a geometric progression:

$$Q_{1\to 2,\text{abs}} = \epsilon P_1 + \epsilon(1-\epsilon)^2 P_1 + \epsilon(1-\epsilon)^4 P_1 + \cdots = \epsilon P_1 \sum_{k=0}^{\infty} [(1-\epsilon)^2]^k$$
$$Q_{1\to 2,\text{abs}} = \frac{\epsilon P_1}{1-(1-\epsilon)^2} = \frac{\epsilon P_1}{2\epsilon - \epsilon^2} = \frac{P_1}{2-\epsilon}$$

The net flux Q_{net} is the difference between the energy transferred from 1 to 2 and from 2 to 1:

$$Q_{\text{net}} = Q_{1\to 2,\text{abs}} - Q_{2\to 1,\text{abs}} = \frac{P_1}{2-\epsilon} - \frac{P_2}{2-\epsilon}$$
$$\boldsymbol{Q_{\text{net}} = \frac{\epsilon \sigma A (T_1^4 - T_2^4)}{2-\epsilon}}$$

(e) The logic is the same as in Part A, but with the new expression for Q_{net}. Relationship between temperatures:

$$T_1^4 - T_{\text{tanque}}^4 = N \cdot \frac{Q_{\text{net}}}{\sigma A}\left(\frac{2-\epsilon}{\epsilon}\right).$$

The energy balance for the outer boundary now includes ϵ:

$$\epsilon A F_0 = \epsilon \sigma A T_1^4 + Q_{\text{net}} \implies T_1^4 = \frac{F_0}{\sigma} - \frac{Q_{\text{net}}}{\epsilon \sigma A} \quad (3)$$

Substituting (3) into the temperature relationship:

$$\left(\frac{F_0}{\sigma} - \frac{Q_{\text{net}}}{\epsilon \sigma A}\right) - T_{\text{tanque}}^4 = N \cdot \frac{Q_{\text{net}}}{\sigma A}\left(\frac{2-\epsilon}{\epsilon}\right)$$
$$\epsilon A F_0 - Q_{\text{net}} - \epsilon \sigma A T_{\text{tanque}}^4 = N \cdot Q_{\text{net}}(2-\epsilon)$$
$$\epsilon A (F_0 - \sigma T_{\text{tanque}}^4) = Q_{\text{net}}[1 + N(2-\epsilon)]$$

Applying the boundary conditions $T_{\text{tanque}} = T_c$ and $Q_{\text{net}} = P_{\text{max}} - P_{\text{int}}$:

$$\epsilon A(F_0 - \sigma T_c^4) = (P_{\text{max}} - P_{\text{int}})[1 + N_{\text{real}}(2-\epsilon)]$$

$$\boldsymbol{N_{\text{real}} = \frac{1}{2-\epsilon}\left[\frac{\epsilon A(F_0 - \sigma T_c^4)}{P_{\text{max}} - P_{\text{int}}} - 1\right]}$$

(f) For $\epsilon \ll 1$, the term $\frac{\epsilon}{2-\epsilon} \approx \frac{\epsilon}{2}$ is much less than 1. Analyzing the expressions, we see that N_{real} is approximately $\frac{\epsilon}{2-\epsilon}$ times N_{ideal}. Therefore, for $\epsilon \ll 1$, **N_{ideal} is significantly larger than N_{real}.**

The use of low-emissivity materials is crucial because they are, as a consequence of Kirchhoff's Law, **highly reflective**. In the black body model ($\epsilon = 1$), all incident energy is absorbed and must be re-radiated, transferring heat efficiently and requiring many shields to create a large temperature gradient. In the realistic model ($\epsilon \ll 1$), when radiation from a hot shield hits the adjacent cooler one, most of that radiation is **reflected** back. This reflection acts as an extremely effective barrier to heat flow. Since the energy transfer per layer is very inefficient, a large temperature difference is easily maintained, and therefore, **far fewer shields are needed** to insulate the cryogenic tank.

Marking Scheme

Part A — 4.0 pts	**Pts**
Apply the Stefan-Boltzmann Law for a surface, $P = \sigma A T^4$	0.2
Express the net flux as the difference between radiated powers, $Q_{\text{net}} = P_i - P_{i+1}$	0.2
Arrive at the correct final expression, $Q_{\text{net}} = \sigma A(T_i^4 - T_{i+1}^4)$	0.2
Isolate the temperature difference, $T_i^4 - T_{i+1}^4 = Q_{\text{net}}/(\sigma A)$	0.2
Write the sum for the N layers, demonstrating understanding of the method	0.6
Correctly identify the cancellation of intermediate terms (telescopic sum)	0.4
Arrive at the correct final relation, $T_1^4 - T_{\text{tanque}}^4 = N \cdot Q_{\text{net}}/(\sigma A)$	0.2
Correctly formulate the energy balance for the first shield (outer boundary): $AF_0 = \sigma A T_1^4 + Q_{\text{net}}$	0.6
Formulate the thermal load condition at the tank (inner boundary): $Q_{\text{net}} = P_{\text{max}} - P_{\text{int}}$	0.6
Substitute the expression for T_1^4 into the equation from item (b)	0.4
Perform the final algebra to isolate and find the correct expression for N_{ideal}	0.4
Part B — 6.0 pts	**Pts**
Identify that the absorbed energy is a sum of terms due to multiple reflections	0.4
Correctly write the first terms of the series for absorbed energy: $\epsilon P_1 + \epsilon(1-\epsilon)^2 P_1 + \ldots$	0.8
Identify the expression as a Geometric Progression and its elements (first term and ratio)	0.6
Correctly calculate the sum of the series for the total absorbed energy from one plate	0.6
Formulate the net flux as the difference of the two energy transfers and arrive at the final expression	0.6
Adapt the telescopic sum logic (item b) for the new Q_{net} formula	0.6
Formulate the new energy balance for the first shield with $\epsilon < 1$: $\epsilon A F_0 = \epsilon\sigma A T_1^4 + Q_{\text{net}}$	0.6
Correctly combine the equations and boundary conditions	0.4
Perform the final algebra to isolate and find the correct expression for N	0.4

Question 2: Coffee Optics - 10 points[2]

When a cup of coffee is illuminated by a lamp or sunlight, it is common to notice bright curved shapes on the surface of the liquid or at the bottom of the cup. These shapes, known as **caustics**, result from the reflection or refraction of light rays, which concentrate in certain regions of space. This is a stunning everyday example of how the geometry of a curved surface can concentrate light, forming patterns that are both aesthetically striking and physically relevant. The study of caustics involves concepts from geometric optics, such as the law of reflection, and is related to classic problems that also appear in advanced optical and astronomical systems.

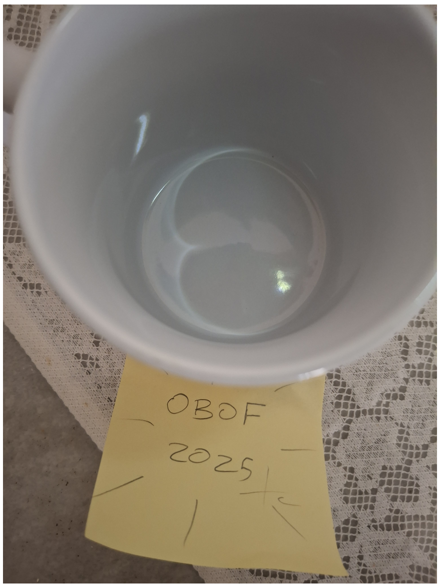

Figure 1: Caustic pattern in a coffee mug.

(A) **6 points** Let C be a small convex mirror with focal length f, and let Γ be a circle with diameter d_1 tangent to C. Suppose P is a point light source located on Γ. Show that the locus of all images of P is another circle Ω, tangent to C, with diameter d_2 satisfying

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}.$$

(B) **4 points** Let P be a fixed point light source and let Γ be a circular mirror of radius R. Show that

- if $P \in \Gamma$, the caustic is a **cardioid**;
- if P is at infinity, the caustic is half of a **nephroid**.

In both cases, these curves can be generated as the path traced by a point on the circumference of a circle that rolls without slipping around another fixed circle of a different radius. Determine the radii of these circles, sketch the curves, and explain why this rolling circle construction does not apply in the general case.

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[2] By Paulo Vinícius

Part A: Locus of the Image

(a) Let $\angle OCP = \theta$, where O is the center of curvature of the mirror C, and let R be a point very close to C such that $\angle RPC = \delta\theta$. Let us denote the image of P by F (see figure).

Since $\triangle POC$ is isosceles, we have

$$\overline{PC} = 2R_1 \cos\theta = d_1 \cos\theta.$$

Let D be the foot of the perpendicular from C onto $\overline{PR}$. For small $\delta\theta$, we can approximate

$$\overline{CD} \approx \overline{PC} \cdot \delta\theta = d_1 \cos\theta\, \delta\theta, \qquad \angle DCR \approx \theta.$$

Hence,

$$\overline{CD} = \overline{CR} \cos\theta \quad \Longrightarrow \quad \overline{CR} = d_1\, \delta\theta.$$

Since $\overline{CR}$ is small and $f = r/2$ for convex mirrors, we have

$$R_O \cdot \angle ROC \approx \overline{CR} \quad \Longrightarrow \quad \angle ROC = \frac{d_1 \delta\theta}{2f}.$$

By symmetry, $\triangle OCR$ is approximately isosceles, so that

$$\angle ORC = \frac{\pi}{2} - \frac{1}{2}\angle ROC = \frac{\pi}{2} - \frac{d_1\delta\theta}{4f}.$$

Let $i = \angle PRO$ be the angle of incidence of the ray $\overline{PR}$. In triangle $\triangle RPC$ we have

$$\begin{cases} \angle RPC = \delta\theta, \\ \angle PRC = \angle ORC - i = \dfrac{\pi}{2} - \dfrac{d_1\delta\theta}{4f} - i, \\ \angle PCR = \angle OCR + \angle PCO = \dfrac{\pi}{2} - \dfrac{d_1\delta\theta}{4f} + \theta. \end{cases}$$

From this relation we conclude

$$i = \theta - \left(\frac{d_1}{2f} - 1\right)\delta\theta.$$

Now let's consider $\triangle CFR$. Its angles are:

$$\begin{cases} \angle CFR = \angle OCR - \angle PCO = \dfrac{\pi}{2} - \dfrac{d_1\delta\theta}{4f} - \theta, \\ \angle FRC = \angle ORC + i = \dfrac{\pi}{2} + \theta - \left(\frac{3d_1}{4f} - 1\right)\delta\theta, \\ \angle CFR = \pi - \angle CFR - \angle FRC = \left(\frac{d_1}{f} - 1\right)\delta\theta. \end{cases}$$

Applying the law of sines in $\triangle CFR$:

$$\frac{\overline{CR}}{\sin(\angle CFR)} = \frac{\overline{CF}}{\sin(\angle FRC)}.$$

Substituting the approximations,

$$\overline{CF} = \frac{d_1 \delta\theta}{\sin\left[\left(\frac{d_1}{f} - 1\right)\delta\theta\right]} \cdot \sin\left[\frac{\pi}{2} + \theta - \left(\frac{3d_1}{4f} - 1\right)\delta\theta\right].$$

Since $\delta\theta$ is small, the last sine is approximately $\cos\theta$, yielding

$$\overline{CF} = \frac{d_1 \delta\theta \cos\theta}{\sin\left[\left(\frac{d_1}{f} - 1\right)\delta\theta\right]}.$$

Using $\sin\phi \approx \phi$ for small ϕ, we have

$$\overline{CF} = \frac{d_1 \cos\theta}{\frac{d_1}{f} - 1} = \frac{\cos\theta}{\frac{1}{f} - \frac{1}{d_1}}.$$

Now construct a point O_2 on $\overline{OC}$ such that $\overline{O_2C} = \overline{O_2F}$. Thus,

$$\overline{O_2C} = \overline{O_2F} = \frac{1}{2\left(\frac{1}{f} - \frac{1}{d_1}\right)},$$

which shows that F belongs to a circle of diameter

$$d_2 = \frac{1}{\frac{1}{f} - \frac{1}{d_1}}.$$

Therefore,

$$\boxed{\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}} \quad \square$$

Part B: Special Caustics

(b) From the result of part (A), we know that the image I of the source P lies at the intersection of a reflected ray and a circle of a certain radius, as illustrated in Figure **??**.

Using the formula obtained in part (A), the circle associated with C has radius r given by

$$\frac{1}{d_\Gamma} + \frac{1}{2r} = \frac{1}{f_\Gamma} \quad \Longrightarrow \quad r = \frac{R}{3}.$$

Since r is constant, we can construct another circle of radius

$$r' = R - 2r = \frac{R}{3},$$

which always remains tangent to the circle containing I.

Now observe the angular relationship:

$$\angle OO'I = 2\angle CO'I = 2\angle PCO = \pi - \angle POC.$$

Since $\angle OO'C + \angle O'OI$ is constant, the angular velocity of O' relative to O coincides with that of I relative to O'. As the two circles have equal radii, the point of contact remains at rest. This shows that the outer circle rolls externally on the fixed circle without slipping, producing the cardioid as the caustic.

In the case where the source is at infinity, the construction changes slightly. On the left side, the caustic does not form due to geometric constraints (see Figure **??**). On the right side, neglecting the $\frac{1}{d_1}$ term in the relation from part (A), we obtain

$$\frac{1}{2r} = \frac{1}{f_\Gamma} \quad \Longrightarrow \quad r = \frac{R}{4}.$$

Thus, the auxiliary circle has radius

$$r' = R - 2r = \frac{R}{2}.$$

By a simple angle notation, we have

$$\angle AOC = \pi - \angle O'CI \quad \Longrightarrow \quad \angle AOC + 2\angle OO'I \text{ is constant.}$$

Since the ratio of the radii is $2:1$, the point of contact remains fixed, which proves that the inner circle rolls without slipping. This construction generates half of a nephroid as the caustic.

In the two special cases ($P \in \Gamma$ or P at infinity), the essential property is that d_1 is constant, allowing the construction with rolling circles. For a general position of P, d_1 varies with θ, so the construction does not work and the caustic ceases to be a standard roulette.

Note: The radius of curvature of the cardioid and the nephroid can be calculated directly by applying kinematic arguments to the rolling circle construction.

Part (A) — 6 pts	**Pts**
Geometric construction of the problem	1.0
Relation $\overline{PC} = d_1 \cos\theta$	1.0
Approximation for small $\delta\theta$ and $\overline{CR} = d_1\delta\theta$	1.0
Calculation of angles of $\triangle CFR$ and other auxiliary angles (only 0.5pt for angles like $\angle OCR$ and $\angle ORC$)	1.0
Law of sines in $\triangle CFR$	1.0
Conclusion $\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$	1.0
Part (B) — 4 pts	**Pts**
Application of the formula from Part (A) for the auxiliary radii	1.0
Geometric construction of the image	1.0
No-slip condition	1.0
Solution for the cardioid	0.3
Solution for the nephroid	0.5
Not mentioning the geometric limitation in the nephroid	-0.3
Comment on why the method fails in the general case	0.2

Question 3: Waves on a String - 10 points[3]

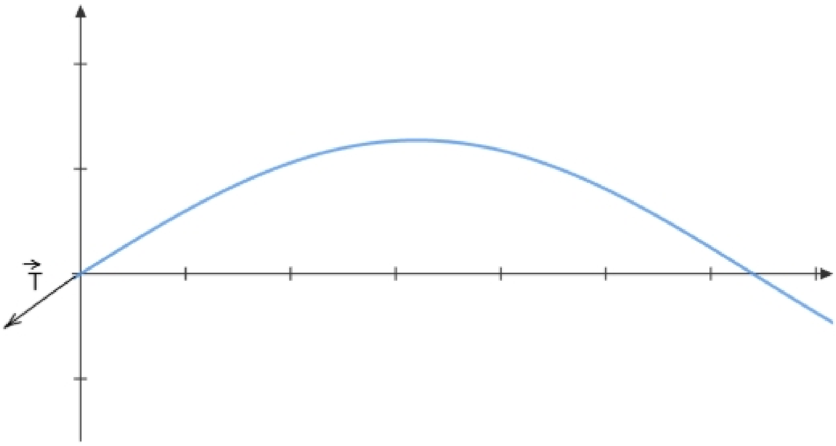

Figure 4: Initial configuration of the string under tension $\vec{T}$.

In the figure above, we have a string with linear mass density ρ which is set to oscillate by a tension T. This force will propagate energy to the string, which will exhibit interesting characteristics. Our intention is to start from a simple case and arrive at a better understanding of waves in general.

Part A: The Wave Equation - 3 points

(a) **2 points** Set up the differential equation of motion for the system.

(b) **1 point** Separate the equation into spatial and temporal parts [Hint: A function $f(x;t)$ can be written as: $\xi(x)\varphi(t)$].

Part B: Energy and Power - 4 points

We will use as a solution:

$$y(x;t) = \mathrm{Re}[Ae^{i(kx-\omega t)}] = A\cos\Big(kx - \omega t + \frac{\pi}{2}\Big)$$
$$y(0;0) = 0 \quad v(0;0) = A\omega$$

(c) **2 points** Calculate the average power of the energy supplied to the string.

(d) **2 points** Express the formulas (in integral form) for the total kinetic and potential energy of the system.

Part C: Principle of Least Action - 3 points

There is a quantity in physics that is useful from general relativity to quantum mechanics, and its name is action, which is defined as:

$$S = \int L(y;\dot{y};t)dt; \quad L = T - V$$

However, when we have systems that, unlike particles, are dispersed in space like strings, we must use $\mathcal{L}$ which is defined as the density of L. We therefore have our new expression:

$$S = \int \mathcal{L}(y;\frac{\partial y}{\partial t};\frac{\partial y}{\partial x};x;t)dxdt$$

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[3] By Yvens Amaral

It is general knowledge that nature evolves to seek the most stable situations, which are those of minimum action. (...) we arrive at the following equation (Euler-Lagrange Equation):

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial(\frac{\partial \mathcal{L}}{\partial \dot{y}})}{\partial t} - \frac{\partial(\frac{\partial \mathcal{L}}{\partial y'})}{\partial x} = 0$$

(Note: $\dot{y} = \partial y/\partial t$ and $y' = \partial y/\partial x$)

(e) **3 points** Using the results from item (d) and the formula above, set up the equation of motion for the system and compare it with the one from item (a).

Part A: The Wave Equation

(a) We apply Newton's 2nd Law ($F_y = ma_y$) to an infinitesimal segment of the string of length Δx and mass $dm = \rho\Delta x$. The net vertical force is the difference in the vertical components of the tension T at the ends. In the small-angle approximation, $T_y \approx T \tan\theta = T\frac{\partial y}{\partial x}$.

$$\begin{aligned} F_{\text{net},y} &= T_y(x+\Delta x) - T_y(x) \\ (dm)a_y &= T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \\ (\rho\Delta x)\frac{\partial^2 y}{\partial t^2} &= T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \end{aligned}$$

Dividing by Δx and taking the limit $\Delta x \to 0$:

$$\begin{aligned} \rho\frac{\partial^2 y}{\partial t^2} &= T \lim_{\Delta x \to 0} \frac{\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right)}{\Delta x} \\ \rho\frac{\partial^2 y}{\partial t^2} &= T\frac{\partial^2 y}{\partial x^2} \implies \boldsymbol{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}} \end{aligned}$$

(b) We assume the separable solution $y(x,t) = \xi(x)\varphi(t)$. We substitute this into the wave equation:

$$\begin{aligned} \frac{\partial^2}{\partial t^2}[\xi(x)\varphi(t)] &= \frac{T}{\rho}\frac{\partial^2}{\partial x^2}[\xi(x)\varphi(t)] \\ \xi(x)\frac{d^2\varphi}{dt^2} &= \frac{T}{\rho}\varphi(t)\frac{d^2\xi}{dx^2} \end{aligned}$$

We divide by $\xi(x)\varphi(t)$ to separate the variables:

$$\underbrace{\frac{1}{\varphi(t)}\frac{d^2\varphi}{dt^2}}_{\text{Function of } t} = \underbrace{\frac{T}{\rho}\frac{1}{\xi(x)}\frac{d^2\xi}{dx^2}}_{\text{Function of } x} = \text{Constant}$$

For oscillatory solutions, the constant must be negative, $-\omega^2$.

Temporal Eq.: $$\frac{d^2\varphi}{dt^2} + \omega^2\varphi(t) = 0$$

Spatial Eq.: $$\frac{d^2\xi}{dx^2} + \left(\frac{\rho\omega^2}{T}\right)\xi(x) = 0$$

Defining $k^2 = \rho\omega^2/T$, the spatial equation becomes $\frac{d^2\xi}{dx^2} + k^2\xi(x) = 0$.

Part B: Energy and Power

(c) The instantaneous power P is the vertical force $F_y = -T\frac{\partial y}{\partial x}$ multiplied by the vertical velocity $v_y = \frac{\partial y}{\partial t}$.

$$P(t) = F_y v_y = \left(-T\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$

Using the solution $y(x,t) = -A\sin(kx - \omega t)$:

$$\frac{\partial y}{\partial x} = -Ak\cos(kx - \omega t)$$
$$\frac{\partial y}{\partial t} = A\omega\cos(kx - \omega t)$$

Substituting into the power:

$$P(t) = -T(-Ak\cos(\cdot))(A\omega\cos(\cdot)) = Tk\omega A^2\cos^2(kx - \omega t)$$

The average power $\bar{P}$ is the time average. Since $\langle\cos^2(\cdot)\rangle = 1/2$:

$$\boldsymbol{\bar{P} = \frac{1}{2}Tk\omega A^2}$$

(Using $T = \rho v^2$ and $k = \omega/v$, also $\bar{P} = \frac{1}{2}\rho v\omega^2 A^2$)

(d) For the Kinetic Energy (E_K), we sum $\frac{1}{2}m_i v_i^2$ and pass to the continuum, where $m_i \to dm = \rho dx$:

$$E_K = \sum \frac{1}{2}m_i\left(\frac{\partial y_i}{\partial t}\right)^2 \quad \longrightarrow \quad \boldsymbol{E_K = \int \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx}$$

For the Potential Energy (E_P), we sum the stretching energy $T \cdot \Delta L$. The stretch $\Delta L \approx \frac{1}{2}(\frac{\partial y}{\partial x})^2 dx$.

$$E_P = \sum T \cdot \Delta L_i \quad \longrightarrow \quad \boldsymbol{E_P = \int \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx}$$

Part C: Principle of Least Action

(e) The Lagrangian density $\mathcal{L}$ is $\mathcal{L} = \mathcal{K} - \mathcal{V}$, where $\mathcal{K}$ and $\mathcal{V}$ are the energy densities (the terms inside the integrals from item (d)).

$$\mathcal{L}(y, \dot{y}, y') = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 - \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \frac{\rho}{2}\dot{y}^2 - \frac{T}{2}(y')^2$$

The Euler-Lagrange Equation is:

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0$$

We calculate each term:

- $\frac{\partial \mathcal{L}}{\partial y} = 0$
- $\frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(\frac{\rho}{2}\dot{y}^2\right) = \rho\dot{y}$

- $\frac{\partial \mathcal{L}}{\partial y'} = \frac{\partial}{\partial y'}\left(-\frac{T}{2}(y')^2\right) = -Ty'$

Substituting into the equation:

$$\begin{aligned}
0 - \frac{\partial}{\partial t}(\rho \dot{y}) - \frac{\partial}{\partial x}(-Ty') &= 0 \\
-\rho \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) + T\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) &= 0 \\
-\rho \frac{\partial^2 y}{\partial t^2} + T\frac{\partial^2 y}{\partial x^2} &= 0 \\
\implies T\frac{\partial^2 y}{\partial x^2} &= \rho \frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

This equation is identical to the Wave Equation derived in Part (a).

Marking Scheme

Part A — 3.0 pts	**Pts**
(a) Apply Newton's 2nd Law and the approximation $T_y \approx T(\partial y/\partial x)$	1.0
(a) Take the limit $\Delta x \to 0$ to arrive at the correct Wave Equation	1.0
(b) Substitute $y = \xi(x)\varphi(t)$ and separate the variables correctly	0.5
(b) Show the two Ordinary Differential Equations (Temporal and Spatial)	0.5
Part B — 4.0 pts	**Pts**
(c) Write the correct formula for instantaneous power $P = F_y v_y$	0.5
(c) Calculate the derivatives $\partial y/\partial x$ and $\partial y/\partial t$ and find $P(t)$	1.0
(c) Calculate the time average $\langle \cos^2 \rangle = 1/2$ and obtain the final $\bar{P}$	0.5
(d) Derive the integral form for Kinetic Energy (E_K)	1.0
(d) Derive the integral form for Potential Energy (E_P)	1.0
Part C — 3.0 pts	**Pts**
(e) Correctly write the Lagrangian density $\mathcal{L} = \mathcal{K} - \mathcal{V}$	1.0
(e) Calculate the partial derivatives $\partial\mathcal{L}/\partial y$, $\partial\mathcal{L}/\partial\dot{y}$, and $\partial\mathcal{L}/\partial y'$	1.0
(e) Substitute into the Euler-Lagrange Equation and show it results in the Wave Eq.	1.0
TOTAL	**10.0**